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Combinatorics and forcing with distributive ideals

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Abstract

We present a version for κ-distributive ideals over a regular infinite cardinal κ of some of the combinatorial results of Mathias on happy families. We also study an associated notion of forcing, which is a generalization of Mathias forcing and of Prikry forcing.

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0. Introduction

A discussion of our results requires some preliminary definitions.

Let κ be a fixed regular infinite cardinal. Throughout this introduction we will take the liberty of confusing any order-type β subset of κ with its increasing enumeration (a sequence of length β).

Let $\boldsymbol{A}$ consist of all strictly increasing $p \in \bigcup_{n\in\omega} \kappa^n$, and let $\boldsymbol{T}$ be a collection of subtrees of the tree $(\boldsymbol{A}, \subseteq)$. Given $a \in [\kappa]^{<\omega}$ and $T \in \boldsymbol{T}$, let $\langle a, T\rangle$ be the set of all $D \in [\kappa]^{\omega}$ such that a is an initial segment of D and $D - a$ is a branch through T. Let $P(\boldsymbol{T})$ be the set consisting of all $\langle a, T\rangle$ for $a \in [\kappa]^{<\omega}$ and $T \in \boldsymbol{T}$. Let $N(\boldsymbol{T})$ be the set of all $W \subseteq [\kappa]^{\omega}$ such that for every $\langle a, T\rangle \in P(\boldsymbol{T})$, there is in $\boldsymbol{T}$ a subtree T' of T such that $\langle a, T'\rangle \cap W = 0$. Members of $N(\boldsymbol{T})$ are said to be *Ramsey null* with respect to $\boldsymbol{T}$. Let $C(\boldsymbol{T})$ be the set of all $W \subseteq [\kappa]^{\omega}$ such that for every $\langle a, T\rangle \in P(\boldsymbol{T})$, there is in $\boldsymbol{T}$ a subtree T' of T such that either $\langle a, T'\rangle \subseteq W$, or else $\langle a, T'\rangle \cap W = 0$. Members of $C(\boldsymbol{T})$ are said to be *completely Ramsey* with respect to $\boldsymbol{T}$.

We recall that a family F of sets is *closed under operation A* if given $W_x \in F$ *for* $x \in \bigcup_{n\in\omega} \omega^n$, we have $\bigcup_{f\in\omega^\omega} \bigcap_{n\in\omega} W_{f|n} \in F$.

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We say that $\boldsymbol{T}$ has the *Mathias property* if the following four conditions are satisfied:

(a) $P(\boldsymbol{T}) \subseteq C(\boldsymbol{T})$;

(b) if $\{\langle a, T\rangle: \langle a, T\rangle \subseteq W$ or $\langle a, T\rangle \cap W = 0\}$ is dense in the partially ordered set $(P(\boldsymbol{T}), \subseteq)$, then $W \in C(\boldsymbol{T})$;

(c) $C(\boldsymbol{T})$ is closed under operation A;

(d) $N(\boldsymbol{T})$ is closed under countable unions.

Notice that the converse of (b) is immediate.

Let us now assume that $0 \notin P(\boldsymbol{T})$, and the intersection of any two members of $P(\boldsymbol{T})$ lies in $P(\boldsymbol{T}) \cup \{0\}$. We put a topology on $[\kappa]^\omega$ by taking 0 and the members of $P(\boldsymbol{T})$ as basic open sets. $\boldsymbol{T}$ is said to have the *Ellentuck property* if the following hold:

(α) $W \in C(\boldsymbol{T})$ if and only if W has the Baire property;

(β) every meager set is nowhere dense.

Given $W \subseteq [\kappa]^\omega$, set $O_W = \bigcup\{\langle a, T\rangle: \langle a, T\rangle \subseteq W\}$. O_W is clearly open. Moreover, $(W - O_W) \cap (O_W \cup O_{[\kappa]^\omega - W}) = 0$. Now if $\{\langle a, T\rangle: \langle a, T\rangle \subseteq W$ or $\langle a, T\rangle \cap W = 0\}$ is dense in $(P(\boldsymbol{T}), \subseteq)$, then the open set $O_W \cup O_{[\kappa]^\omega - W}$ is dense, and therefore W has the Baire property. Thus every $W \in C(\boldsymbol{T})$ has the Baire property, and every $W \in N(\boldsymbol{T})$ is nowhere dense (since then $O_W = 0$). Let us show that $\boldsymbol{T}$ has the Mathias property if and only if $\boldsymbol{T}$ has the Ellentuck property. First assume that $\boldsymbol{T}$ has the Mathias property. By (b), every open set lies in $C(\boldsymbol{T})$. It easily follows that $[\kappa]^\omega - O \in N(\boldsymbol{T})$ for every dense open set O. Hence every nowhere dense set lies in $N(\boldsymbol{T})$, and by (d), so does every meager set. Thus if W has the Baire property, then there is an $O \in C(\boldsymbol{T})$ such that $(O - W) \cup (W - O) \in N(\boldsymbol{T})$ and, consequently, $W \in C(\boldsymbol{T})$. Hence (β) and (α) both hold. Now assume that $\boldsymbol{T}$ has the Ellentuck property. From (α) one easily derives (b), (c) and since every open set has the Baire property, (a). Every nowhere dense set has the Baire property and, hence, by (α), lies in $N(\boldsymbol{T})$. Now applying (β), we obtain (d).

In the following $\mathfrak{I}$ will denote an ideal over κ such that $\kappa \subseteq \mathfrak{I} \subset P(\kappa)$. We set $\boldsymbol{T}(\mathfrak{I}) = \{T_C: C \in \mathfrak{I}^+\}$, where $T_C = \{p \in A: ran(p) \subset C\}$. We let $\boldsymbol{T}'(\mathfrak{I})$ be the set of all subtrees T of $\boldsymbol{A}$ such that for every $n \in \omega$ and every $p \in T \cap \kappa^n$, $\{q(n): q \in T$ and $p \subset q\} \in \mathfrak{I}^*$ (i.e. such that the set of immediate successors of any node of T lies in $\mathfrak{I}^*$).

Let us for the time being assume that $\kappa = \omega$. Working with the topology on $[\omega]^\omega$ obtained by taking 0 and the $\langle a, T_\omega\rangle$ for $a \in [\omega]^{<\omega}$ as basic open sets, Galvin and Prikry [10] proved that every Borel set lies in $C(\boldsymbol{T}([\omega]^{<\omega}))$. So does every analytic set, by a result of Silver [29]. Ellentuck [9] showed that $\boldsymbol{T}([\omega]^{<\omega})$ has the Ellentuck property (whence the name), thus giving a new proof of Silver's theorem. Assuming that $\mathfrak{I}$ is prime and weakly selective (i.e. that $\mathfrak{I}^*$ is a Ramsey ultrafilter), Mathias [24] proved that $\boldsymbol{T}(\mathfrak{I})$ has the Mathias property (in fact $\boldsymbol{T}(\mathfrak{I})$ has the slicker Ellentuck property, which was shown by Louveau [21] and Milliken [26]). Generalizing his result (see [25] for a historical account), Mathias [24] proved that $\boldsymbol{T}(\mathfrak{I})$ has the Mathias property whenever $\mathfrak{I}^+$ is a happy family (i.e. whenever $\mathfrak{I}$ is weakly selective and $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$ is $\aleph_0$-closed). Let us point out that as Mathias himself observed, $\mathfrak{I}^+$ is a happy family in case $\mathfrak{I} = [\omega]^{<\omega}$. Louveau showed in [22] that $\boldsymbol{T}'(\mathfrak{I})$ has the Ellentuck property whenever $\mathfrak{I}$ is prime (this is actually a straightforward reformulation of his result).

Our original goal was to obtain a result that would be to the result of Louveau on prime ideals what the result of Mathias on happy families is to the result of Mathias on weakly selective prime ideals. What kind of ideals should be considered? What we are looking for is a nontrivial property satisfied by every prime ideal. Let us examine the conditions on $\mathfrak{I}$ for $\mathfrak{I}^+$ to be a happy family, and let us try to find an adequate weakening. It is natural to drop weak selectivity, and so we are left with $\aleph_0$-closedness. We settled for the closely related, but weaker, $\aleph_0$-distributivity. The next task was to find the appropriate $\boldsymbol{T}$. Let us define $\bar{\boldsymbol{T}}(\mathfrak{I})$ as the set of all subtrees T of $\boldsymbol{A}$ such that the following hold:

(0) setting $C = \{p(0)\colon p \in T - \{0\}\}$, $C \in \mathfrak{I}^+$ and for every $n \in \omega$ and every $p \in T$ with $dom(p) = n$, $C - \{q(n)\colon q \in T \text{ and } p \subset q\} \in \mathfrak{I}$;

(1) for every $n > 0$ and every $p \in T$ with $dom(p) = n$,

$$\{q(n)\colon q \in T \text{ and } p \subset q\} \subseteq \{r(n-1)\colon r \in T \text{ and } r|(n-1) = p|(n-1)\}.$$

So a member of $\bar{\boldsymbol{T}}(\mathfrak{I})$ is a subtree of some T_C obtained by applying to each node an operation consisting in deleting some of its immediate successors. It is easily verified that in case $\mathfrak{I}$ is prime, $C(\bar{\boldsymbol{T}}(\mathfrak{I})) = C(\boldsymbol{T}'(\mathfrak{I}))$ and $N(\bar{\boldsymbol{T}}(\mathfrak{I})) = N(\boldsymbol{T}'(\mathfrak{I}))$. We show that $\bar{\boldsymbol{T}}(\mathfrak{I})$ has the Mathias property whenever $\mathfrak{I}$ is $\aleph_0$-distributive. As a byproduct, we obtain the following improvement of the result of Mathias: $\boldsymbol{T}(\mathfrak{I})$ has the Mathias property whenever $\mathfrak{I}$ is $\aleph_0$-distributive and weakly selective.

Let us now switch from combinatorics to forcing. Mathias [24] studied forcing with the partially ordered set $(P(\boldsymbol{T}(\mathfrak{I})), \subseteq)$, under the assumption that $\mathfrak{I}^+$ is a happy family (in the literature the expression 'Mathias forcing' usually refers to that notion of forcing in either one of the following two special cases: (i) $\mathfrak{I} = [\omega]^{<\omega}$; (ii) $\mathfrak{I}$ is prime and weakly selective). As for the notion of forcing $(P(\boldsymbol{T}'(\mathfrak{I})), \subseteq)$, where $\mathfrak{I}$ is prime, it has been studied by Blass [4], and more recently by Judah and Shelah [13, 16]. Either notion of forcing is known to preserve ω_1, to adjoin to the universe a dominating (in fact Ramsey) real, to produce many degrees of constructibility, and to satisfy the following property: Given any condition $\langle a, T\rangle$ and any sentence φ of the forcing language, there is a stronger condition $\langle a, T'\rangle$ that decides φ. We establish the same facts for the notion of forcing $(P(\bar{\boldsymbol{T}}(\mathfrak{I})), \subseteq)$, where $\mathfrak{I}$ is $\aleph_0$-distributive. Forcing with $(P(\bar{\boldsymbol{T}}(\mathfrak{I})), \subseteq)$ is equivalent to forcing with $(P(\boldsymbol{T}(\mathfrak{I})), \subseteq)$ in case $\mathfrak{I}$ is $\aleph_0$-distributive and weakly selective, and to forcing with $(P(\boldsymbol{T}'(\mathfrak{I})), \subseteq)$ in case $\mathfrak{I}$ is prime. In fact, forcing with $(P(\bar{\boldsymbol{T}}(\mathfrak{I})), \subseteq)$ is the same as forcing first with $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$, which adds a prime ideal $J \supseteq \mathfrak{I}$, and then forcing with $(P(\boldsymbol{T}'(J)), \subseteq)$. This decompostion result is similar to that of [24].

It is well-known that there is a close analogy between forcing with $(P(\boldsymbol{T}(\mathfrak{I})), \subseteq)$ and Prikry forcing, which is defined using a normal κ-complete prime ideal over κ. It appeared to us that it might be interesting to make explicit that similarity by treating the two notions in the same framework. That meant extending our study to arbitrary κ's, which turned out to require little extra work. Assuming that $\mathfrak{I}$ is κ-distributive and κ-complete, we show that the notion of forcing $(P(\bar{\boldsymbol{T}}(\mathfrak{I})), \subseteq)$ adds no new bounded subsets of κ, preserves κ^+, satisfies the $(\kappa^+ \cdot sat(\mathfrak{I}))$-chain condition and adjoins to the universe an order-type ω cofinal subset of κ. It is actually equivalent to Prikry forcing in case $\mathfrak{I}$ is κ-complete, normal and prime.

As for the combinatorial side, we show that $\bar{T}(\mathfrak{I})$ has the Mathias property whenever $\mathfrak{I}$ is κ-distributive. As every prime ideal is trivially κ-distributive, our result is no empty shell for uncountable κ's. Still, there are in case $\kappa = \omega$ plenty of κ-distributive ideals that are nowhere prime. Are there any such ideals in case κ is uncountable? Some answers can fortunately be found in the literature. Johnson [14] and Levinski [20] have each constructed examples of such objects (starting from a measurable cardinal). Their ideals are κ-complete (and satisfy further properties). Besides Levinski [20] has shown that it is consistent for the least cardinal κ where the Generalized Continuum Hypothesis fails to bear an ideal of that type. Such κ's are known not to be measurable. More examples can be found in Section 2. We show for instance that under Martin's axiom, there is a ν-distributive nowhere prime ideal over any regular uncountable cardinal $\nu < 2^{\aleph_0}$. There are however many cardinals (e.g. $2^{\aleph_0}$) for which we do not know whether they can bear such ideals.

We chose to formulate our results in terms of games. Consider the two player infinite game played as follows: I picks $A_0 \in \mathfrak{I}^+$; II answers by selecting $\alpha_0 \in A_0$; I picks $A_1 \subseteq A_0$ with $A_0 - A_1 \in \mathfrak{I}$; II answers by selecting $\alpha_1 \in A_1$ with $\alpha_1 > \alpha_0$; etc. Then there is an obvious one-to-one correspondence between strategies for player I and members of $\bar{T}(\mathfrak{I})$. Granted, there is no obvious gain in clarity in restating our results in terms of those games. The point is that our proofs are based on other, more complicated games, which were already used by Kastanas (in the case $\mathfrak{I} = [\omega]^{<\omega}$), in his analysis [17] of Ellentuck's theorem. Moreover we give a characterization of $C(\bar{T}(\mathfrak{I}))$ in terms of determinacy of the latter games. Our approach leads us to work with a partially ordered set $(P_{\mathfrak{I}}, \leqslant)$ whose definition is not as nice as that of $(P(\bar{T}(\mathfrak{I})), \subseteq)$.

Section 1 reviews some basic material on ideals. In Section 2, we recall elementary facts concerning distributivity properties of ideals. Section 3 introduces almost κ-distributivity, a natural weakening of κ-distributivity. This paper is to a large extent a study of the properties of (almost) κ-distributive ideals (even though some results, that are proved under the assumption of κ-distributivity of the ideal, could conceivably be in fact valid for all $(\kappa, 2)$-distributive ideals). Let us however point out that we failed to produce any example of an almost κ-distributive ideal that is not κ-distributive. Let us also remark that the existence of a σ-complete almost κ-distributive ideal is a large cardinal property. In fact, let J be such an ideal. Then by Proposition 3.3, J is $add(J)$-distributive. Hence by Propositions 1.1 and 14.4, $add(J)$ is a Ramsey cardinal. Section 4 is devoted to Kastanas games, with a focus on the role of player I. Proposition 4.3 makes explicit the connection with Ramsey theory.

Sections 5 and 6 are, respectively, devoted to $N_{\mathfrak{I}}$ (our version of the nowhere Ramsey (also called Ramsey null) sets) and $C_{\mathfrak{I}}$ (our analog of the completely Ramsey sets). Assuming κ uncountable, Propositions 5.7 and 5.5 tell us that the degree of completeness of the ideal $N_{\mathfrak{I}}$ is easily ascertained provided that $\mathfrak{I}$ is σ-complete (and almost κ-distributive), or that the Continuum Hypothesis holds. We do not know whether there are non σ-complete ideals J over κ such that N_J is $\aleph_2$-complete. Let $\mathfrak{t}$ denote the least infinite cardinal ν such that $([\omega]^\omega/[\omega]^{<\omega}, \leqslant)$ is not ν-closed. Then there is (by results of Section 2 and Proposition 10.21) for each regular uncountable cardinal $\mu \leqslant \mathfrak{t}$, a weakly selective nowhere prime ideal J over ω with $add(N_J) = \mu$. That is a nowhere prime version of a result of Louveau [22], which states that under Martin's axiom, there is for each regular uncountable cardinal $\mu \leqslant 2^{\aleph_0}$, a weakly selective prime ideal J over ω with $add(N_{\mathfrak{J}}) = \mu$. Let us remark that except for Proposition 5.7 results in those two sections are obtained without imposing any distributivity requirement on $\mathfrak{J}$.

Two results that are essential for the remainder of the paper are featured in Section 7: the reduction result of Proposition 7.1 that asserts that a winning strategy for player II can be used to define a winning strategy for player I in the complementary game, and the technical Proposition 7.3 that establishes an analog of the 'capturing' method used by Mathias in his paper [24] on happy families. Let us comment in passing on our partial order $\leqslant$ on $P_{\mathfrak{J}}$. Its definition can hardly be claimed to be highly intuitive. It however makes the later proofs go through, whereas the more natural $\subseteq$ seems unmanageable. Section 8 presents the combinatorial results we were aiming at: characterizations of $N_{\mathfrak{J}}$ and $C_{\mathfrak{J}}$, and closure of $C_{\mathfrak{J}}$ under operation A. Proposition 8.5 is however not totally satisfactory, as one would like to have the stronger closure property of Proposition 9.7. This better result would easily be derived if only we could show that $P_{\mathfrak{J}} \subseteq C_{\mathfrak{J}}$. Our failure to do so in a general way (it is plain that $P_{\mathfrak{J}} \subseteq C_{\mathfrak{J}}$ in case $\mathfrak{J}$ is prime, and also in case $\kappa = \omega$ and $\mathfrak{J}$ is nowhere tall) casts an unpleasant shadow on the picture. After all if it were to run out that $P_{\mathfrak{J}} - C_{\mathfrak{J}} \neq 0$, then the whole approach taken in this paper (i.e. working with $P_{\mathfrak{J}}$ rather than with $P_{\mathfrak{J}}^*$) could be argued to be the wrong one.

The reformulation of the results of Section 8 in the next section, assuming a bit more, should make them more palatable. This stronger assumption, requiring as it does that $\mathfrak{J}$ be either κ-distributive, or else almost κ-distributive and σ-complete, looks however somewhat awkward. Using the notation above, we have $N_{\mathfrak{J}} = N(\bar{\boldsymbol{T}}(\mathfrak{J}))$ by Proposition 9.3 and $C_{\mathfrak{J}} = C(\bar{\boldsymbol{T}}(\mathfrak{J}))$ by Proposition 9.6. A comparison between Propositions 8.3 and 9.6 indicates that $C_{\mathfrak{J}}^*$ might be worth studying, where $C_{\mathfrak{J}}^*$ denotes the set of all $W \subseteq [\kappa]^\omega$ such that $G_{\mathfrak{J}}^*(a, C, W)$ is determined for all a, C. (Let us observe that under the hypotheses of Proposition 9.6, one easily gets $C_{\mathfrak{J}} \subseteq C_{\mathfrak{J}}^*$). A similar remark can be made in connection with Proposition 10.11. One would expect the hypothesis of Corollary 9.4 to imply that $\boldsymbol{h}_{\mathfrak{J}} \geqslant add(N_{\mathfrak{J}})$, but we have been unable to derive that much. In Section 10, we present new characterizations of $N_{\mathfrak{J}}$ and $C_{\mathfrak{J}}$ under the assumption that $\mathfrak{J}$ is a κ-distributive weak P-point. We also compute $add(N_{\mathfrak{J}})$, $cov(N_{\mathfrak{J}}), non(N_{\mathfrak{J}})$ and $cof(N_{\mathfrak{J}})$ is some special cases. In [7] Brendle calculates these four cardinal coefficients in case $\kappa = \omega$ and $\mathfrak{J}$ is prime, and some of our results (namely all those that involve the cardinals $\nu_{\mathfrak{J}}$, $\chi_{\mathfrak{J}}$ and $\boldsymbol{m}_{\mathfrak{J}}$) were obtained after seeing his paper. Propositions 10.20 and 10.23 generalize the corresponding results of Mathias [24] for happy families.

The next four sections are devoted to the study of the notion of forcing $(P_{\mathfrak{I}}, \leqslant)$. We remark that Propositions 11.7 and 11.8 required some more work to be established than the corresponding facts for Mathias or Prikry forcing. The fact is that the notion of forcing $(P_{\mathfrak{I}}^*, \subseteq)$ is much easier to handle, which explains why the proof of Proposition 12.11 involves an awkward detour by $P_{\mathfrak{I}}^*$. We have been unable to describe adequately the situation with respect to cardinal collapsing. Notice that by Proposition 14.3 a closer study of the notion of forcing $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$ could do a lot to clarify the case. By Propositions 14.3, 14.5 and 14.6, forcing with $(P_{\mathfrak{I}}, \leqslant)$ can be decomposed as a two stage iteration in case $\mathfrak{I}$ is κ-distributive and κ-complete. Let us remark that by an argument of Brendle [6], we have the following. Assume $\kappa = \omega$ and $\mathfrak{I}$ is $\aleph_0$-distributive, and let $x \in [\kappa]^\omega$ be $P_{\mathfrak{I}}$-generic over V. Then in $V[x]$, $\boldsymbol{d} = \omega_1$.

The brief Section 15 introduces the game $G_{\mathfrak{I}}^{**}$. In Section 16 we restrict our attention to ideals over ω. We see Proposition 16.2 as bearing witness to the naturalness of the notion of an $\aleph_0$-distributive weakly selective ideal. Proposition 16.11 exhibits a hierarchical list of properties that can be additionally satisfied by an $\aleph_0$-distributive ideal. Well-known (see [12, 3]) specific ideals are used to show the hierarchy to be strict. Numerous problems are left open. It would for instance be interesting to determine which ideals J are such that for every $C \in J^+$, II has no winning strategy in $G_J^*(0, C, J^+)$ (respectively in $G_J^{**}(0, C, J^+)$). One should also find out whether there is any ZFC example of an ideal J such that the game $G_J^{**}(0, \omega, J \cap [\omega]^\omega)$ is undetermined.

1. Ideals

This section is devoted to the notion of ideal (over a set). We recall basic definitions and review some elementary facts that will be needed later. Proofs will often be omitted.

Let X be an infinite set. Given $K \subseteq P(X)$, we define $add(K)$ as follows. In case $\bigcup S \in K$ for every $S \subseteq K$, set $add(K) = (2^{|X|})^+$. Otherwise let $add(K)$ be the smallest cardinality of any $S \subseteq K$ with $\bigcup S \notin K$.

Given $J \subseteq P(X)$, J is an *ideal over* X if the following three conditions are satisfied:
(i) $P(A) \subseteq J$ for every $A \in J$; (ii) $A \cup B \in J$ whenever $A, B \in J$;
(iii) $\{x\} \in J$ for every $x \in X$.

Let J be an ideal over X. We set $J^+ = P(X) - J$ and $J^* = \{A \subseteq X : X - A \in J\}$.

For each $A \in J^+$, we put $J \mid A = \{B \subseteq X\colon A \cap B \in J\}$.

J is *prime* if $J^+ \subseteq J^*$.

J is *nowhere prime* if for each $A \in J^+$, $J \mid A$ is not prime.

It is easy to see that J is nowhere prime if and only if for every $A \in J^+$, there is a $B \in J^+ \cap P(A)$ with $A - B \in J^+$.

Let $J \subset P(X)$ be an ideal over X. We let $cov(J)$ be the smallest cardinality of any $S \subseteq J$ with $\bigcup S = X$.

$add(J)$ is easily seen to be a regular infinite cardinal. Clearly, $add(J) \leqslant cov(J) \leqslant |X|$. Moreover, $add(J) = cov(J)$ in case J is prime. Notice that for every $A \in J^+$, $add(J \mid A) \geqslant add(J)$ and $cov(J \mid A) \leqslant cov(J)$.

We will sometimes (for instance in Sections 12 and 14) have to assume that J verifies $add(J) = cov(J)$. The assumption is not very restrictive, as the following folklore result shows.

Proposition 1.1. *Let $J \subset P(X)$ be an ideal over X. Then there is an $A \in J^+$ such that $add(J) = add(J \mid A) = cov(J \mid A)$.*

Proof. Select $A_\alpha \in J$ for $\alpha < add(J)$ with $\bigcup_{\alpha < add(J)} A_\alpha \in J^+$, and put $A = \bigcup_{\alpha < add(J)} A_\alpha$. Clearly $A_\alpha \in J \mid A$ for every α. As $X = (X - A) \cup \bigcup_{\alpha < add(J)} A_\alpha$, we have

$cov(J \mid A) \leqslant add(J)$. Hence $add(J) = add(J \mid A) = cov(J \mid A)$. $\square$

Let $J \subset P(X)$ be an ideal over X. We let $non(J)$ be the smallest cardinality of any $W \subseteq X$ with $W \notin J$.

Observe that $non(J) \leqslant |X|$ and $add(J) \leqslant cof(non(J))$.

We let $cof(J)$ be the smallest cardinality of any $S \subseteq J$ with $J = \bigcup_{W \in S} P(W)$.

One easily shows that $cof(J) \geqslant non(J)$ and $cof(cof(J)) \geqslant add(J)$. Notice that for every $A \in J^+$, $non(J \mid A) \geqslant non(J)$ and $cof(J \mid A) \leqslant cof(J)$.

Given a set A of ordinals and an ordinal γ, we set $[A]^\gamma = \{B \subseteq A\colon o.t.\,(B) = \gamma\}$ and $[A]^{<\gamma} = \bigcup_{\delta \in \gamma} [A]^\delta$.

We let e_A denote the increasing enumeration of A (i.e. the unique isomorphism of $(o.t.\,(A), \in)$ onto $(A, \in)$).

Throughout the remainder of this paper, κ and $\mathfrak{I}$ will, respectively, denote a regular infinite cardinal, and an ideal over κ with $\kappa \subseteq \mathfrak{I} \subset P(\kappa)$.

We recall that $\boldsymbol{b}_\kappa$ (respectively $\boldsymbol{d}_\kappa$) is the least cardinality of any $F \subseteq \kappa^\kappa$ with the following property: given $f \in \kappa^\kappa$, $\{\alpha\colon f(\alpha) \leqslant g(\alpha)\} \in [\kappa]^\kappa$ (resp. $\{\alpha\colon f(\alpha) > g(\alpha)\} \in [\kappa]^{<\kappa}$) for some $g \in F$.

We set $\boldsymbol{b} = \boldsymbol{b}_\omega$ and $\boldsymbol{d} = \boldsymbol{d}_\omega$.

Given $A \in \mathfrak{I}^+$, we let $D_{\mathfrak{I},A}$ be the set of all $Q \subseteq \mathfrak{I}^+ \cap P(A)$ such that $B \cap C \in \mathfrak{I}$ whenever B, C are distinct members of Q.

Thus a member of $D_{\mathfrak{I},A}$ is a family of almost disjoint mod $\mathfrak{I}$ subsets of A.

We let $M_{\mathfrak{I},A}$ be the set of all $Q \in D_{\mathfrak{I},A}$ such that for each $B \in \mathfrak{I}^+ \cap P(A)$, there is a $C \in Q$ with $B \cap C \in \mathfrak{I}^+$.

Notice that $M_{\mathfrak{I},A}$ consists of all maximal elements of the partially ordered set $(D_{\mathfrak{I},A}, \subseteq)$.

Given $Q, Q' \in M_{\mathfrak{I},A}$, we let $Q' \leqslant Q$ if and only if for every $C \in Q'$, there exists $B \in Q$ with $C - B \in \mathfrak{I}$.

$sat(\mathfrak{I})$ is the least cardinal λ such that $|Q| < \lambda$ for every $Q \in D_{\mathfrak{I},\kappa}$.

It is well-known that $sat([\kappa]^{<\kappa}) > \boldsymbol{b}_\kappa$.

For each $A \subseteq \kappa$, we put $[A]_{\mathfrak{I}} = \{B \subseteq \kappa\colon (A - B) \cup (B - A) \in \mathfrak{I}\}$.

We set $\mathfrak{I}^+/\mathfrak{I} = \{[A]_{\mathfrak{I}}\colon A \in \mathfrak{I}^+\}$.

Given $A, B \in \mathfrak{I}^+$, we let $[A]_{\mathfrak{I}} \leqslant [B]_{\mathfrak{I}}$ just in case $A - B \in \mathfrak{I}$.

Let $(P, \leqslant)$ be a partially ordered set, and let μ be an infinite cardinal. Then $(P, \leqslant)$ is *μ-closed* if the following holds: let ν be an infinite cardinal $\leqslant \mu$, and let $p_\alpha \in P$ for $\alpha < \nu$ be such that $p_\alpha \leqslant p_\beta$ for all $\beta < \alpha$. Then there is a $p \in P$ such that $p \leqslant p_\alpha$ for all $\alpha < \nu$.

The following is well-known.

Proposition 1.2. *Assume that $\mathfrak{I}$ is nowhere prime, and let μ be an infinite cardinal such that $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$ admits a μ-closed dense subset. Then $sat(\mathfrak{I} \mid A) > 2^\mu$ for every $A \in \mathfrak{I}^+$.*

Proof. Left to the reader. $\square$

$\mathfrak{I}$ is *tall* if $\mathfrak{I} \cap [A]^\kappa \neq 0$ for every $A \in \mathfrak{I}^+$.

$\mathfrak{I}$ is *nowhere tall* if for each $A \in \mathfrak{I}^+$, $\mathfrak{I} \mid A$ is not tall.

Notice that $\mathfrak{I}$ is nowhere tall if and only if for every $A \in \mathfrak{I}^+$, there is a $B \in \mathfrak{I}^+ \cap P(A)$ with $[B]^\kappa \subseteq \mathfrak{I}^+$.

$\mathfrak{I}$ is *feeble* if there is an $f \in \kappa^\kappa$ such that $\{f^{-1}(\{\alpha\})\colon \alpha \in \kappa\} \subseteq [\kappa]^{<\kappa}$ and $\{f^{-1}(E)\colon E \in [\kappa]^\kappa\} \subseteq \mathfrak{I}^+$.

Notice that if $\mathfrak{I}$ is not tall, then $\mathfrak{I}$ is feeble. The following is immediate.

Proposition 1.3. *Assume that $\mathfrak{I}$ is feeble. Then $sat(\mathfrak{I}) \geqslant sat([\kappa]^{<\kappa})$.*

The following is well-known (see [30, 2, 5] for the case $\kappa = \omega$).

Lemma 1.4. *The following are equivalent:*

(i) *$\mathfrak{I}$ is feeble.*

(ii) *There is an increasing $f \in \kappa^\kappa$ such that $\{f^{-1}(E)\colon E \in [\kappa]^\kappa\} \subseteq \mathfrak{I}^+$.*

(iii) *There exists $g \in \kappa^\kappa$ such that for every $A \in \mathfrak{I}^*$, $\{\alpha\colon e_A(\alpha) \geqslant g(\alpha)\} \in [\kappa]^{<\kappa}$.*

Proof. (i) $\to$ (ii). Assume that $\mathfrak{I}$ is feeble, and let $g \in \kappa^\kappa$ be such that $\{g^{-1}(\{\alpha\})\colon \alpha \in \kappa\} \subseteq [\kappa]^{<\kappa}$ and $\{g^{-1}(E)\colon E \in [\kappa]^\kappa\} \subseteq \mathfrak{I}^+$. Select $\alpha_\beta \in \kappa$ for $\beta < \kappa$ so that $g^{-1}(\{\alpha_\beta\}) \neq 0$ and for all $\gamma < \beta$, $\bigcap g^{-1}(\{\alpha_\beta\}) > \bigcup g^{-1}(\{\alpha_\gamma\})$. Now define $f \in \kappa^\kappa$ so that every $\beta < \kappa, f^{-1}(\{\beta\}) = \{\delta \in \kappa\colon \bigcap g^{-1}(\{\alpha_\beta\}) \leqslant \delta < \bigcap g^{-1}(\{\alpha_{\beta+1}\})\}$. Then f is as desired.

(ii) $\to$ (iii): Let f be as in the statement of (ii). Define $g \in \kappa^\kappa$ by letting $g(\gamma) = \bigcap f^{-1}(\{\gamma + \gamma + 1\})$ in case $f^{-1}(\{\gamma + \gamma + 1\}) \neq 0$, and $g(\gamma) = 0$ otherwise. Now fix $A \in \mathfrak{I}^*$. Let $\beta \in \kappa$ be such that $A \cap f^{-1}(\{\alpha\}) \neq 0$ whenever $\beta \leqslant \alpha < \kappa$. For every $\delta \in \kappa$, we have $f(e_A(\beta + \delta)) \leqslant \beta + \beta + \delta \leqslant \beta + \delta + \beta + \delta$ and therefore $e_A(\beta + \delta) < g(\beta + \delta)$.

(iii) $\to$ (i): Let g be as in the statement of (iii), and define $h \in \kappa^\kappa$ by letting $h(\alpha) = (\bigcup g[\alpha + 1]) + 1$. Then define η_α for $\alpha < \kappa$ so that $\eta_0 = 0$, $\eta_{\alpha+1} = h(\eta_\alpha)$ and in case α is a limit >0, $\eta_\alpha = \bigcup_{\beta < \alpha} \eta_\beta$. Now define $f \in \kappa^\kappa$ by letting $f^{-1}(\{\alpha\}) = \{\gamma \in \kappa\colon \eta_\alpha \leqslant \gamma < \eta_{\alpha+1}\}$. Fix $E \in [\kappa]^\kappa$, and set $A = \kappa - \bigcup f^{-1}(E)$. If $\kappa - E \in [\kappa]^{<\kappa}$, then clearly $A \in \mathfrak{I}$. Thus assume that $\kappa - E \in [\kappa]^\kappa$. Define $q \in \kappa^E$ by letting $q(\delta) = \bigcap((\kappa - E) - \delta)$. Observe that $ran(q) \in [\kappa]^\kappa$. Given $\delta \in E$, set $\gamma = e_A^{-1}(\eta_{q(\delta)})$. Then clearly $\eta_\delta \geqslant \gamma$. Hence $e_A(\gamma) \geqslant \eta_{\delta+1} \geqslant h(\gamma) \geqslant g(\gamma)$. Therefore, $A \notin \mathfrak{I}^*$. □

Lemma 1.5. *Let $A \in \mathfrak{I}^+$ and $f \in \kappa^\kappa$. Then there exists $C \in \mathfrak{I}^+ \cap P(A)$ such that $\{\alpha: e_C(\alpha) \geqslant f(\alpha)\} \in [\kappa]^\kappa$.*

Proof. First assume that $\mathfrak{I} \mid A$ is not feeble. Then by Lemma 1.4, there is a $D \in (\mathfrak{I} \mid A)^*$ such that $\{\alpha: e_D(\alpha) \geqslant f(\alpha)\} \in [\kappa]^\kappa$. Now setting $C = D \cap A$, we have that C is as desired. Next assume that $\mathfrak{I} \mid A$ is feeble. Then by Lemma 1.4, there is an increasing $g \in \kappa^A$ such that for every $E \in [\kappa]^\kappa$, $\bigcup g^{-1}(E) \in \mathfrak{I}^+$. Define $\alpha_\beta \in \kappa$ for $\beta \in \kappa$ so that

(0) $g^{-1}(\{\alpha_\beta\}) \neq 0$;
(1) $\alpha_\gamma < \alpha_\beta$ whenever $\gamma < \beta$;
(2) $\bigcap g^{-1}(\{\alpha_0\}) \geqslant f(0)$;
(3) $\bigcap g^{-1}(\{\alpha_\beta\}) \geqslant f(o.t.(\bigcup_{\gamma<\beta} g^{-1}(\{\alpha_\gamma\})))$ for $\beta > 0$.

Then set $C = \bigcup_{\beta<\kappa} g^{-1}(\{\alpha_\beta\})$. C is clearly as desired. □

It is easy to see that $\mathfrak{I}$ is feeble if and only if $\mathfrak{I} \mid A$ is feeble for some $A \in \mathfrak{I}^+$. This motivates the following definition.

$\mathfrak{I}$ is *everywhere feeble* if $\mathfrak{I} \mid A$ is feeble for every $A \in \mathfrak{I}^+$.

$\mathfrak{I}$ is a *weak P-point* if given $A \in \mathfrak{I}^+$ and $f: A \to \kappa$ such that $f^{-1}(\{\alpha\}) \in \mathfrak{I}$ for all $\alpha < \kappa$, there is a $B \in \mathfrak{I}^+ \cap P(A)$ such that $|B \cap f^{-1}(\{\alpha\})| < \kappa$ for all $\alpha < \kappa$.

It is easy to see that if $\mathfrak{I}$ is a weak P-point, then $add(\mathfrak{I}) = \kappa$. We will now introduce what appears to be a natural measure of the degree of weak P-pointness of $\mathfrak{I}$.

We define $\pi_\mathfrak{I}$ as follows. Let us first assume that there exists $X \subseteq \mathfrak{I}$ with the following property: There is an $A \in \mathfrak{I}^+$ such that for every $C \in \mathfrak{I}^+ \cap P(A)$, $\{C \cap B: B \in X\} \cap [\kappa]^\kappa \neq 0$. Then we let $\pi_\mathfrak{I}$ be the least cardinality of any such X. Otherwise we set $\pi_\mathfrak{I} = (2^\kappa)^+$.

We observe that $\pi_\mathfrak{I} \leqslant \pi_{\mathfrak{I} \mid A}$ for every $A \in \mathfrak{I}^+$.

The following is readily verified.

Proposition 1.6. (i) *$\pi_\mathfrak{I} = (2^\kappa)^+$ if and only if $\pi_\mathfrak{I} > cof(\mathfrak{I})$ if and only if $\mathfrak{I}$ is nowhere tall.*
(ii) *If $\mathfrak{I}$ is not a weak P-point, then $\pi_\mathfrak{I} = add(\mathfrak{I})$.*

The following is folklore.

Proposition 1.7. *Assume $\mathfrak{I}$ is a weak P-point. Then $\pi_\mathfrak{I} > \kappa$.*

Proof. Let $A \in \mathfrak{I}^+$ and $H_\alpha \in \mathfrak{I}$ for $\alpha < \kappa$. The desired conclusion is immediate in case $A \cap \bigcup_{\alpha<\kappa} H_\alpha \in \mathfrak{I}$. So let us assume that $A \cap \bigcup_{\alpha<\kappa} H_\alpha \in \mathfrak{I}^+$. Define $f: A \cap \bigcup_{\alpha<\kappa} H_\alpha \to \kappa$ by letting $f(\gamma) = \delta$ just in case $\gamma \in H_\delta - \bigcup_{\alpha<\delta} H_\alpha$. Now select

$C \in \mathfrak{I}^+ \cap P(A \cap \bigcup_{\alpha<\kappa} H_\alpha)$ so that $|C \cap f^{-1}(\{\delta\})| < \kappa$ for all $\delta < \kappa$. By regularity of κ, we clearly have that $|C \cap H_\alpha| < \kappa$ for every $\alpha < \kappa$. □

$\mathfrak{I}$ is a *weak Q-point* if given $A \in \mathfrak{I}^+$ and $f: A \to \kappa$ such that $f^{-1}(\{\alpha\}) \in [A]^{<\kappa}$ for all $\alpha \in \kappa$, there is a $C \in \mathfrak{I}^+ \cap P(A)$ with f being one-to-one on C.

The following is folklore.

Lemma 1.8. *The following are equivalent:*

(i) *$\mathfrak{I}$ is a weak Q-point.*

(ii) *Given $A \in \mathfrak{I}^+$ and $H_\alpha \in [\kappa]^{<\kappa}$ for $\alpha \in A$, there is a $C \in \mathfrak{I}^+ \cap P(A)$ such that $\beta \notin \bigcup_{\alpha \in C \cap \beta} H_\alpha$ for all $\beta \in C$.*

Proof. (i) → (ii): Assume (i), and let $A \in \mathfrak{I}^+$ and $H_\alpha \in [\kappa]^{<\kappa}$ for $\alpha \in A$. Define G_γ and K_γ for $\gamma \in \kappa$ by letting $K_\gamma = ((\gamma \cup \bigcup_{\alpha \in G_\gamma} (\alpha \cup H_\alpha)) \cap A) - G_\gamma$, $G_0 = \{0\}$ and $G_\gamma = \bigcup_{\beta<\gamma} K_\beta$ in case $\gamma > 0$. Set $E_0 = \bigcup \{K_{\eta+2n}$: η is a limit ordinal $<\kappa$ and $n \in \omega\}$ and $E_1 = A - E_0$. Pick $i < 2$ so that $E_i \in \mathfrak{I}^+$. Then select $C \in \mathfrak{I}^+ \cap P(E_i)$ so that for every $\gamma \in \kappa$, $|C \cap K_\gamma| \leqslant 1$. Now fix $\alpha, \beta \in C$ with $\alpha < \beta$. Let $\gamma, \zeta \in \kappa$ be such that $\alpha \in K_\gamma$ and $\beta \in K_\zeta$. Then $\beta \in G_{\zeta+1}$ and therefore $\alpha \in G_{\zeta+2}$. As clearly $\gamma \notin \{\zeta, \zeta+1\}$, we have that $\gamma + 2 \leqslant \zeta$. It follows that $\beta \notin H_\alpha$, since $A \cap H_\alpha \subseteq G_{\gamma+2}$.

(ii) → (i): Assume (ii), and let $A \in \mathfrak{I}^+$ and $f: A \to \kappa$ be such that $f^{-1}(\{\delta\}) \in [A]^{<\kappa}$ for all $\delta \in \kappa$. Pick $C \in \mathfrak{I}^+ \cap P(A)$ so that $\beta \notin \bigcup_{\alpha \in C \cap \beta} f^{-1}(\{f(\alpha)\})$ for all $\beta \in C$. Clearly, f is one-to-one on C. □

$\mathfrak{I}$ is a *weak semi-Q-point* if given $A \in \mathfrak{I}^+$ and $f \in \kappa^\kappa$, there is a $C \in \mathfrak{I}^+ \cap P(A)$ such that for every $\alpha \in \kappa$, $e_C(\alpha + 1) \geqslant f(\alpha)$.

It easily follows from Lemma 1.8 that if $\mathfrak{I}$ is a weak Q-point, then $\mathfrak{I}$ is a weak semi-Q-point.

$\mathfrak{I}$ is *weakly selective* (respectively *weakly semiselective*) if $\mathfrak{I}$ is both a weak Q-point (resp. a weak semi-Q-point) and a weak P-point.

We recall that $\mathfrak{I}^+ \to (\mathfrak{I}^+)^2$ asserts the following. Given $W \subseteq [\kappa]^2$ and $A \in \mathfrak{I}^+$, there is a $C \in \mathfrak{I}^+ \cap P(A)$ such that either $[C]^2 \subseteq W$, or else $[C]^2 \cap W = 0$.

We conclude the section with a study of the cardinal $\nu_{\mathfrak{I}}$, which will be used in Sections 5 and 10.

We let $\nu_{\mathfrak{I}}$ be the least cardinality of any $W \subseteq [\kappa]^\omega$ such that for every $Q \in M_{\mathfrak{I},\kappa}$, $\{A - B$: $A \in W$ and $B \in Q\} \cap [\kappa]^{<\omega} \neq 0$.

We observe that $\nu_{\mathfrak{I}} \leqslant \nu_{\mathfrak{I}|A}$ for every $A \in \mathfrak{I}^+$.

Proposition 1.9. *Assuming $\kappa = \omega$, $\nu_{\mathfrak{I}} \geqslant$ **b**.*

Proof. Let $W \subseteq [\omega]^\omega$ be such that $0 < |W| <$ **b**. Pick $f \in \omega^\omega$ so that for every $A \in W$, $\{n$: $f(n) \leqslant e_A(2n)\} \in [\omega]^{<\omega}$. By Lemma 1.5 there is a $Q \in M_{\mathfrak{I},\omega}$ such that for every $B \in Q$, $\{n$: $e_B(n) \geqslant f(n)\} \in [\omega]^\omega$. Now suppose there are $B \in Q$ and $A \in W$ such that $A - B \in [\omega]^{<\omega}$. Then there is an $r \in \omega$ such that for every $m \in \omega$, $e_A(r + m) \geqslant e_B(m)$.

Select $m \geqslant r$ so that $e_B(m) \geqslant f(m) > e_A(2m)$. Then $e_A(2m) \geqslant e_A(r+m) > e_A(2m)$, a contradiction. $\square$

Proposition 1.10. $\nu_{\mathfrak{I}} \geqslant \kappa$.

Proof. The assertion is immediate from Proposition 1.9 in case $\kappa = \omega$. Thus assume that $\kappa > \omega$, and let $W \subset [\kappa]^\omega$ be such that $|W| < \kappa$. Setting $Q = \{\kappa - \bigcup W\}$, we have that $Q \in M_{\mathfrak{I},\kappa}$ and $\{A - B\colon A \in W \text{ and } B \in Q\} \cap [\kappa]^{<\omega} = 0$. $\square$

Proposition 1.11. *Assume that $\kappa = \omega$ and $\mathfrak{I}$ is a weak semi-Q-point. Then $\nu_{\mathfrak{I}} \geqslant \boldsymbol{d}$.*

Proof. Let $W \subseteq [\omega]^\omega$ be such that $0 < |W| < \boldsymbol{d}$. Pick $f \in \omega^\omega$ so that for every $A \in W$, $\{n\colon f(n) > e_A(2n)\} \in [\omega]^\omega$. There is a $Q \in M_{\mathfrak{I},\omega}$ such that for every $B \in Q$ and every $n \in \omega$, $e_B(n+1) \geqslant f(n)$. Now suppose there are $B \in Q$ and $A \in W$ such that $A - B \in [\omega]^{<\omega}$. Then there is an $r \in \omega$ such that for every $m \in \omega$, $e_A(r+m) \geqslant e_B(m)$. Select $m > 0$ so that $f(r+m) > e_A(2(r+m))$. We have

$$e_B(r+m+1) \geqslant f(r+m) > e_A(2(r+m)) \geqslant e_B(m+r+m) \geqslant e_B(1+r+m),$$

a contradiction. $\square$

Proposition 1.12. *Assume that $\kappa = \omega$ and $\mathfrak{I}$ is tall. Then $\nu_{\mathfrak{I}} \geqslant \pi_{\mathfrak{I}}$.*

Proof. Let $W \subseteq [\omega]^\omega$ be such that $0 < |W| < \pi_{\mathfrak{I}}$. For each $A \in W$, pick $B_A \in \mathfrak{I} \cap [A]^\omega$. Now there is a $Q \in M_{\mathfrak{I},\omega}$ such that for every $D \in Q$, $\{D \cap B_A\colon A \in W\} \subseteq [\omega]^{<\omega}$. Clearly, $\{A - D\colon A \in W \text{ and } D \in Q\} \subseteq [\omega]^\omega$. $\square$

Proposition 1.13. *Assume that $\kappa = \omega$ and $\mathfrak{I}$ is everywhere feeble. Then $\nu_{\mathfrak{I}} = 2^{\aleph_0}$.*

Proof. Let $W \subseteq [\omega]^\omega$ be such that $0 < |W| < 2^{\aleph_0}$. Given $C \in \mathfrak{I}^+$, there is an $R \in D_{\mathfrak{I},C} \cap D_{[\omega]^{<\omega},C}$ with $|R| = 2^{\aleph_0}$. Then there exists $D \in R$ such that for each $A \in W$, $A - D \notin [\omega]^{<\omega}$. It follows that there is a $Q \in M_{\mathfrak{I},\omega}$ such that for every $A \in W$, $\{A - B\colon B \in Q\} \subseteq [\omega]^\omega$. $\square$

2. Distributive ideals

In this section we recall basic definitions, and prove some elementary facts, about distributivity properties of ideals over κ.

Let μ, η be cardinals $\geqslant 1$. $\mathfrak{I}$ is *(μ,η)-distributive* if the following holds. Let $Q_\alpha \in M_{\mathfrak{I},\kappa}$ for $\alpha < \mu$ with $|Q_\alpha| \leqslant \eta$, and let $B \in \mathfrak{I}^+$. Then there are $C \in \mathfrak{I}^+ \cap P(B)$ and $h \in \prod_{\alpha<\mu} Q_\alpha$ such that for every $\alpha < \mu$, $C - h(\alpha) \in \mathfrak{I}$.

Observe that for every cardinal $\mu \geqslant 1$, $\mathfrak{I}$ is (μ,μ)-distributive if and only if $\mathfrak{I}$ is $(\mu, 2)$-distributive. Johnson [15] showed that $\mathfrak{I}^+ \to (\mathfrak{I}^+)^2$ if and only if $\mathfrak{I}$ is $(\kappa, 2)$-distributive and weakly selective. The following is well-known.

Proposition 2.1. *Let μ, η be cardinals $\geqslant 1$. Then the following are equivalent:*

(i) *$\mathfrak{I}$ is (μ,η)-distributive.*

(ii) *Let $A \in \mathfrak{I}^+$, and let $Q_\alpha \in M_{\mathfrak{I},A}$ for $\alpha < \mu$ with $|Q_\alpha| \leqslant \eta$. Then there is a $Q \in M_{\mathfrak{I},A}$ such that for every $\alpha < \mu$, $Q \leqslant Q_\alpha$.*

(iii) *$\mathfrak{I} | A$ is (μ,η)-distributive for every $A \in \mathfrak{I}^+$.*

Proof. (i) $\to$ (ii): Assume that (i) holds. Let $A \in \mathfrak{I}^+$, and let $Q_\alpha \in M_{\mathfrak{I},A}$ for $\alpha < \mu$ with $|Q_\alpha| \leqslant \eta$. Select $k \in \prod_{\alpha<\mu} Q_\alpha$, and for each $\alpha < \mu$, put

$$R_\alpha = \{(\kappa - A) \cup k(\alpha)\} \cup (Q_\alpha - \{k(\alpha)\}).$$

Clearly each $R_\alpha \in M_{\mathfrak{I},\kappa}$. Hence by (i), there is an $R \in M_{\mathfrak{I},\kappa}$ such that for every $\alpha < \mu$, $R \leqslant R_\alpha$. Put $Q = \{B \cap A\colon B \in R\}$. Then $Q \in M_{\mathfrak{I},A}$. Moreover, $Q \leqslant Q_\alpha$ for all $\alpha < \mu$.

(ii) $\to$ (iii): Assume that (ii) holds. Let $Q_\alpha \in M_{\mathfrak{I}|A,\kappa}$ for $\alpha < \mu$ with $|Q_\alpha| \leqslant \eta$. For each $\alpha < \mu$, put $R_\alpha = \{B \cap A\colon B \in Q_\alpha\}$. Clearly each $R_\alpha \in M_{\mathfrak{I},A}$. By (ii) there is an $R \in M_{\mathfrak{I},A}$ such that for every $\alpha < \mu$, $R \leqslant R_\alpha$. Given $B \in (\mathfrak{I}|A)^+$, select $C \in R$ with $C \cap B \in \mathfrak{I}^+$. Let $h \in \prod_{\alpha<\mu} Q_\alpha$ be such that for each $\alpha < \mu$, $C - h(\alpha) \in \mathfrak{I}$. Clearly, $(C \cap B) - h(\alpha) \in \mathfrak{I}|A$ for all $\alpha < \mu$.

(iii) $\to$ (i): Trivial. $\square$

Let $(P, \leqslant)$ be a partially ordered set, and let μ be a cardinal $\geqslant 1$. Then $(P, \leqslant)$ is *μ-distributive* if $\bigcap_{\alpha<\mu} W_\alpha$ is dense in $(P, \leqslant)$ whenever each W_α is dense and open in $(P, \leqslant)$.

We omit the proof of the following, which is folklore.

Lemma 2.2. *Given a cardinal $\mu \geqslant 1$, $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$ is μ-distributive if and only if $\mathfrak{I}$ is $(\mu, 2^\kappa)$-distributive.*

Given a cardinal $\mu \geqslant 1$, $\mathfrak{I}$ is *μ-distributive* if $\mathfrak{I}$ is $(\mu, 2^\kappa)$-distributive.

We define $\boldsymbol{s}_\mathfrak{I}$ (respectively $\boldsymbol{h}_\mathfrak{I}$) as follows. In case $\mathfrak{I}$ is $(\mu,2)$-distributive (resp. μ-distributive) for every cardinal $\mu \geqslant 1$, we set $\boldsymbol{s}_\mathfrak{I} = (2^\kappa)^+$ (resp. $\boldsymbol{h}_\mathfrak{I} = (2^\kappa)^+$). Otherwise we let $\boldsymbol{s}_\mathfrak{I}$ (resp. $\boldsymbol{h}_\mathfrak{I}$) be the least cardinal $\mu \geqslant 1$ such that $\mathfrak{I}$ is not $(\mu,2)$-distributive (resp. not μ-distributive).

We put $\boldsymbol{s} = \boldsymbol{s}_{[\omega]^{<\omega}}$.

It is well-known that $\boldsymbol{s}_\mathfrak{I} \cap \boldsymbol{\pi}_\mathfrak{I} \leqslant \boldsymbol{s}_{[\kappa]^{<\kappa}} \leqslant \boldsymbol{d}_\kappa$. Assuming that $add(\mathfrak{I}) = \kappa$, it is readily seen that for any cardinal $v < \kappa$, $v < \boldsymbol{s}_\mathfrak{I}$ if and only if $2^v < \kappa$.

Clearly, $\boldsymbol{h}_\mathfrak{I}$ is a regular infinite cardinal with $\boldsymbol{h}_\mathfrak{I} \leqslant cof(\boldsymbol{s}_\mathfrak{I})$. The easy proof of the following is left to the reader.

Lemma 2.3. *$\mathfrak{I}|A$ is nowhere prime for some $A \in \mathfrak{I}^+$ if and only if $\boldsymbol{h}_{\mathfrak{I}} \leqslant 2^\kappa$ if and only if $\boldsymbol{s}_{\mathfrak{I}} \leqslant 2^\kappa$.*

We remark that for every $A \in \mathfrak{I}^+$, $\boldsymbol{s}_{\mathfrak{I}} \leqslant \boldsymbol{s}_{\mathfrak{I}|A}$ and $\boldsymbol{h}_{\mathfrak{I}} \leqslant \boldsymbol{h}_{\mathfrak{I}|A}$.

Proposition 2.4. *Assume $\mathfrak{I}$ is nowhere tall. Then $\boldsymbol{h}_{\mathfrak{I}} = \boldsymbol{h}_{[\kappa]^{<\kappa}}$.*

Proof. Let us first show that $\boldsymbol{h}_{\mathfrak{I}} \geqslant \boldsymbol{h}_{[\kappa]^{<\kappa}}$. Thus let $X \subseteq M_{\mathfrak{I},\kappa}$ and $E \in \mathfrak{I}^+$ be such that for every $D \in \mathfrak{I}^+ \cap P(E)$, there is a $Q \in X$ with $\{D - B: B \in Q\} \subseteq \mathfrak{I}^+$. Pick $C \in [E]^\kappa$ such that $[C]^\kappa \subseteq \mathfrak{I}^+$. Define $g: X \to P(P(\kappa))$ by letting $g(Q) = (\{B \cap C: B \in Q\} \cup \{\kappa - C\}) \cap [\kappa]^\kappa$. It is readily checked that $g(Q) \in M_{[\kappa]^{<\kappa},\kappa}$. Clearly, for every $D \in [C]^\kappa$, there is a $Q \in X$ with $\{D - A: A \in g(Q)\} \subseteq [\kappa]^\kappa$.

We next show that $\boldsymbol{h}_{[\kappa]^{<\kappa}} \geqslant \boldsymbol{h}_{\mathfrak{I}}$. Thus let $Y \subseteq M_{[\kappa]^{<\kappa},\kappa}$ and $A \in [\kappa]^\kappa$ be such that for every $D \in [A]^\kappa$, there is a $Q \in Y$ with $\{D - B: B \in Q\} \subseteq [\kappa]^\kappa$. Select $H \in \mathfrak{I}^+$ with $[H]^\kappa \subseteq \mathfrak{I}^+$. Let $j: \kappa \to H$ be one-to-one and onto. For every $Q \in Y$, put $S_Q = \{j[B]: B \in Q\} \cup (\{\kappa - H\} \cap \mathfrak{I}^+)$. Clearly, $S_Q \in M_{\mathfrak{I},\kappa}$. Moreover, for every $D \in \mathfrak{I}^+ \cap P(j[A])$, there is a $Q \in Y$ with $\{D - T: T \in S_Q\} \subseteq \mathfrak{I}^+$. □

Given an infinite cardinal μ, $\mathfrak{I}$ *is nowhere μ-distributive* if for every $A \in \mathfrak{I}^+$, $\mathfrak{I}|A$ is not μ-distributive.

Proposition 2.5. *Let μ be an infinite cardinal. Then the following are equivalent:*

(i) *$\mathfrak{I}$ is nowhere μ-distributive.*

(ii) *There are $Q_\alpha \in M_{\mathfrak{I},\kappa}$ for $\alpha < \mu$ such that for every $C \in \mathfrak{I}^+$ and every $h \in \prod_{\alpha<\mu} Q_\alpha$, $\{C - h(\alpha): \alpha < \mu\} \cap \mathfrak{I}^+ \neq 0$.*

Proof. Left to the reader. □

Corollary 2.6. *Assume $\boldsymbol{h}_{\mathfrak{I}} \leqslant 2^\kappa$. Then there is an $A \in \mathfrak{I}^+$ such that $\boldsymbol{h}_{\mathfrak{I}|A} = \boldsymbol{h}_{\mathfrak{I}}$ and $\mathfrak{I}|A$ is nowhere $\boldsymbol{h}_{\mathfrak{I}}$-distributive.*

Proof. Left to the reader. □

We now introduce some more cardinals associated with $\mathfrak{I}$. The forthcoming definition of $\boldsymbol{k}_{\mathfrak{I}}$ may seem less natural than that of $\boldsymbol{h}_{\mathfrak{I}}$ (which measures the degree of distributivity of $\mathfrak{I}$). Moreover, $\boldsymbol{k}_{\mathfrak{I}}$ can easily be computed from two other cardinals that were defined earlier. We feel however that it is worth defining because of its important role in Section 10 (see Corollary 10.5 and Proposition 10.21).

We let $\boldsymbol{k}_{\mathfrak{I}}$ be the least cardinality of any $X \subseteq M_{\mathfrak{I},\kappa}$ with the following property: There is a $C \in \mathfrak{I}^+$ such that for every $D \in \mathfrak{I}^+ \cap P(C)$, $\{D - B: B \in Q\} \subseteq [\kappa]^\kappa$ for some $Q \in X$.

$\boldsymbol{k}_{\mathfrak{I}}$ is readily seen to be a regular infinite cardinal. The following proposition collects some more easy facts concerning $\boldsymbol{k}_{\mathfrak{I}}$.

Proposition 2.7. (0) $\boldsymbol{k}_{\mathfrak{I}} \leqslant cof(\pi_{\mathfrak{I}})$.

(1) $\boldsymbol{k}_{\mathfrak{I}} = \boldsymbol{h}_{\mathfrak{I}} \cap \pi_{\mathfrak{I}}$.

(2) *If $\mathfrak{I}$ is prime, then $\boldsymbol{k}_{\mathfrak{I}} = \pi_{\mathfrak{I}}$.*

(3) $\boldsymbol{k}_{\mathfrak{I}} \leqslant cof(\nu_{\mathfrak{I}})$ *in case* $\kappa = \omega$.

We let $\chi_{\mathfrak{I}}$ be the least cardinality of any $X \subseteq M_{\mathfrak{I},\kappa}$ such that for every $E \in [\kappa]^\kappa$, there is a $Q \in X$ with $\{E - B\colon B \in Q\} \subseteq [\kappa]^\kappa$.

It is easily checked that $\boldsymbol{k}_{\mathfrak{I}} \leqslant \chi_{\mathfrak{I}} \leqslant \boldsymbol{s}_{[\kappa]^{<\kappa}}$ and in case $cov(\mathfrak{I}) < \kappa$, $\chi_{\mathfrak{I}} \leqslant cov(\mathfrak{I})$. Notice that $\chi_{\mathfrak{I}|A} \leqslant \chi_{\mathfrak{I}}$ for every $A \in \mathfrak{I}^+$.

Proposition 2.8. $\chi_{\mathfrak{I}} \leqslant \boldsymbol{d}_\kappa$.

Proof. Let $F \subseteq \kappa^\kappa$ be such that each $f \in F$ is increasing and for every $g \in \kappa^\kappa$, there is an $f \in F$ with $\{\alpha\colon g(\alpha) > f(\alpha)\} \in [\kappa]^{<\kappa}$. Given $f \in F$, there is by Lemma 1.5 a $Q_f \in M_{\mathfrak{I},\kappa}$ such that for all $D \in Q_f$, $\{\alpha\colon e_D(\alpha) \geqslant f(\alpha + \alpha) + 1\} \in [\kappa]^\kappa$. Now fix $C \in [\kappa]^\kappa$. Suppose that $f \in F$ and $D \in Q_f$ are such that $C - D \in [\kappa]^{<\kappa}$. Put $\delta_f = \bigcap \{\zeta \in \kappa\colon e_C[\kappa - \zeta] \subseteq D\}$ and $Z_f = \{\alpha\colon e_D(\alpha) \geqslant f(\alpha + \alpha) + 1\}$. Then for every $\eta \in Z_f \cap (\kappa - \delta_f)$, $e_C(\delta_f + \eta) \geqslant e_D(\eta) > f(\eta + \eta) \geqslant f(\delta_f + \eta)$. Thus, $(\alpha\colon e_C(\alpha) > f(\alpha)\} \in [\kappa]^\kappa$. Hence there is an $f \in F$ such that for every $D \in Q_f$, $C - D \in [\kappa]^\kappa$. $\square$

Proposition 2.9. *Assume $\mathfrak{I}$ is a weak semi-Q-point. Then* $\chi_{\mathfrak{I}} \leqslant \boldsymbol{b}_\kappa$.

Proof. Let $F \subseteq \kappa^\kappa$ be such that each $f \in F$ is increasing and for every $g \in \kappa^\kappa$, there is an $f \in F$ with $\{\alpha\colon f(\alpha) > g(\alpha)\} \in [\kappa]^\kappa$. Given $f \in F$, select $Q_f \in M_{\mathfrak{I},\kappa}$ so that $e_D(\alpha + 1) \geqslant f(\alpha + \alpha)$ for all $D \in Q_f$ and $\alpha < \kappa$. Now fix $C \in [\kappa]^\kappa$. Suppose that $f \in F$ and $D \in Q_f$ are such that $C - D \in [\kappa]^{<\kappa}$. Put $\delta_f = \bigcap \{\zeta \in \kappa\colon e_C[\kappa - \zeta] \subseteq D\}$. Then for every $\eta \in \kappa - \delta_f$, $e_C(\delta_f + \eta + 1) \geqslant e_D(\eta + 1) \geqslant f(\eta + \eta) \geqslant f(\delta_f + \eta)$. Thus $\{\alpha\colon e_C(\alpha + 1) < f(\alpha)\} \in [\kappa]^{<\kappa}$. Hence there is an $f \in F$ such that for every $D \in Q_f$, $C - D \in [\kappa]^\kappa$. $\square$

Proposition 2.10. *Assume $\mathfrak{I}$ is nowhere tall. Then* $\chi_{\mathfrak{I}} = \boldsymbol{k}_{\mathfrak{I}} = \boldsymbol{h}_{\mathfrak{I}}$.

Proof. We have $\boldsymbol{k}_{\mathfrak{I}} = \boldsymbol{h}_{\mathfrak{I}}$ by Propositions 2.7 and 1.6. As $\boldsymbol{k}_{\mathfrak{I}} \leqslant \chi_{\mathfrak{I}}$, there only remains to show that $\chi_{\mathfrak{I}} \leqslant \boldsymbol{h}_{\mathfrak{I}}$. $\mathfrak{I}$ is nowhere $\boldsymbol{h}_{\mathfrak{I}}$-distributive by Proposition 2.4. Hence by Proposition 2.5 there are $Q_\alpha \in M_{\mathfrak{I},\kappa}$ for $\alpha < \boldsymbol{h}_{\mathfrak{I}}$ such that for every $C \in \mathfrak{I}^+$ and every $h \in \prod_{\alpha < \boldsymbol{h}_{\mathfrak{I}}} Q_\alpha$, $\{C - h(\alpha)\colon \alpha < \boldsymbol{h}_{\mathfrak{I}}\} \cap \mathfrak{I}^+ \neq 0$. For each $\alpha < \boldsymbol{h}_{\mathfrak{I}}$, select

$$F_\alpha \in \prod_{B \in Q_\alpha} (M_{\mathfrak{I},B} \cap \{D \in [\kappa]^\kappa\colon [D]^\kappa \subseteq \mathfrak{I}^+\})$$

and set $R_\alpha = \bigcup_{B \in Q_\alpha} F_\alpha(B)$. Notice that $R_\alpha \in M_{\mathfrak{I},\kappa}$. Clearly given $E \in [\kappa]^\kappa$ and $h \in \prod_{\alpha < \boldsymbol{h}_{\mathfrak{I}}} R_\alpha$, we have that

$$\{E - h(\alpha)\colon \alpha < \boldsymbol{h}_{\mathfrak{I}}\} \cap [\kappa]^\kappa \neq 0. \qquad \square$$

We let $\boldsymbol{m}_{\mathfrak{J}}$ be the least cardinality of any $X \subseteq M_{\mathfrak{J},\kappa}$ such that for every $Q \in M_{\mathfrak{J},\kappa}$, there exists $R \in X$ with the following property: for each $A \in R$, there is a $B \in Q$ with $A - B \in [\kappa]^{<\omega}$.

We remark that $\boldsymbol{m}_{\mathfrak{J}|A} \leqslant \boldsymbol{m}_{\mathfrak{J}}$ for every $A \in \mathfrak{J}^+$.

Proposition 2.11. $\boldsymbol{m}_{\mathfrak{J}} \geqslant sat(\mathfrak{J})$.

Proof. Let $Q \in M_{\mathfrak{J},\kappa}$, and let $g: Q \to M_{\mathfrak{J},\kappa}$ be one-to-one. Given $C \in \mathfrak{J}^+$ and $S \in M_{\mathfrak{J},\kappa}$, it is easy to find $D \in \mathfrak{J}^+ \cap P(C)$ and $E \in S$ such that $D \subseteq E$ and $E - D \in [\kappa]^\kappa$. Hence for each $A \in Q$, there exists $R_A \in M_{\mathfrak{J},A}$ with the following property: given $D \in R_A$, there exists $E \in g(A)$ such that $D \subseteq E$ and $E - D \in [\kappa]^\kappa$. Now set $R = \bigcup_{A \in Q} R_A$. Clearly $R \in M_{\mathfrak{J},\kappa}$. Moreover, for every $A \in Q$, there is an $E \in g(A)$ with $\{E - D: D \in R\} \subseteq [\kappa]^\kappa$. □

The following is straightforward.

Proposition 2.12. (0) *If $\mathfrak{J}$ is prime, then $\boldsymbol{m}_{\mathfrak{J}} = cof(\mathfrak{J})$.*

(1) $\boldsymbol{m}_{\mathfrak{J}} \geqslant \nu_{\mathfrak{J}}$.

(2) *$\boldsymbol{m}_{\mathfrak{J}} \geqslant \boldsymbol{h}_{\mathfrak{J}}$ in case $\boldsymbol{h}_{\mathfrak{J}} \leqslant 2^\kappa$.*

(3) *Assuming $\kappa = \omega$, $cof(\boldsymbol{m}_{\mathfrak{J}}) \geqslant \boldsymbol{k}_{\mathfrak{J}}$.*

Proposition 2.13. *Assume $\mathfrak{J}$ is nowhere tall. Then $\boldsymbol{m}_{\mathfrak{J}} = \boldsymbol{m}_{[\kappa]^{<\kappa}}$.*

Proof. Let us first show that $\boldsymbol{m}_{\mathfrak{J}} \leqslant \boldsymbol{m}_{[\kappa]^{<\kappa}}$. Thus let $X \subseteq M_{[\kappa]^{<\kappa},\kappa}$ be such that for every $P \in M_{[\kappa]^{<\kappa},\kappa}$, there is an $R \in X$ with $R \subseteq \{E \cup a: E \in P \text{ and } a \in [\kappa]^{<\omega}\}$. Given $R \in X$, put $S_R = R \cap \mathfrak{J}^+$. Given $C \in \mathfrak{J}^+$, pick $D \in [C]^\kappa$ with $[D]^\kappa \subseteq \mathfrak{J}^+$. Then select $A \in R$ with $A \cap D \in [\kappa]^\kappa$. Clearly, $A \cap C \in \mathfrak{J}^+$ and $A \in S_R$. Hence, $S_R \in M_{\mathfrak{J},\kappa}$. Now fix $Q \in M_{\mathfrak{J},\kappa}$. Set $T = \{C \cap B: C \in Z \text{ and } B \in Q\} \cap \mathfrak{J}^+$, where $Z \in M_{\mathfrak{J},\kappa}$ is such that $\bigcup_{C \in Z} [C]^\kappa \subseteq \mathfrak{J}^+$. Clearly $T \in M_{\mathfrak{J},\kappa} \cap D_{[\kappa]^{<\kappa},\kappa}$. Now select $P \in M_{[\kappa]^{<\kappa},\kappa}$ and $R \in X$ so that $T \subseteq P$ and $R \subseteq \{E \cup a: E \in P \text{ and } a \in [\kappa]^{<\omega}\}$. Then for every $A \in S_R$, there is a $B \in Q$ with $A - B \in [\kappa]^{<\omega}$.

Let us next show that $\boldsymbol{m}_{[\kappa]^{<\kappa}} \leqslant \boldsymbol{m}_{\mathfrak{J}}$. Thus let $X \subseteq M_{\mathfrak{J},\kappa}$ be such that for every $Q \in M_{\mathfrak{J},\kappa}$, there is an $R \in X$ with $R \subseteq \{B \cup a: B \in Q \text{ and } a \in [\kappa]^{<\omega}\}$. Now fix $E \in \mathfrak{J}^+$ such that $[E]^\kappa \subseteq \mathfrak{J}^+$. For each $R \in X$, set $P_R = \{E \cap A: A \in R\} \cap [\kappa]^\kappa$ and $S_R = \{e_E^{-1}(H): H \in P_R\}$. It is readily checked that $S_R \in M_{[\kappa]^{<\kappa},E}$. Now given $T \in M_{[\kappa]^{<\kappa},E}$, set $Q = \{e_E[C]: C \in T\} \cup (\{\kappa - E\} \cap \mathfrak{J}^+)$. Then $Q \in M_{\mathfrak{J},\kappa}$ and therefore there is an $R \in X$ with $R \subseteq \{B \cup a: B \in Q \text{ and } a \in [\kappa]^{<\omega}\}$. Given $D \in S_R$, there is an $A \in R$ such that $e_E[D] = E \cap A$. Let $B \in Q$ be such that $A - B \in [\kappa]^{<\omega}$. We have $B = e_E[C]$, where $C \in T$. Clearly, $D - C \in [\kappa]^{<\omega}$. □

We will next show that if certain conditions are satisfied, then the following holds: For every regular infinite cardinal $\mu \leqslant \boldsymbol{h}_{\mathfrak{J}}$, there exists an ideal J over κ that satisfies $\mathfrak{J} \subseteq J$ and $\boldsymbol{h}_J = \mu$.

For each $Q \in M_{\mathfrak{I},\kappa}$, we put $\Theta_Q = \{C \in \mathfrak{I}^+ : |\{B \in Q : C \cap B \in \mathfrak{I}^+\}| = 2^\kappa\}$.

The following is similar to Lemma 1.12 in [1].

Proposition 2.14. *Assume that $\mathfrak{I}$ is nowhere $\boldsymbol{h}_{\mathfrak{I}}$-distributive, and let $Z \subseteq \mathfrak{I}^+/\mathfrak{I}$ be dense in $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$. Then for each cardinal ρ with $1 \leqslant \rho < \bigcap_{A \in \mathfrak{I}^+} sat(\mathfrak{I} \mid A)$, there are $Q_\alpha \in M_{\mathfrak{I},\kappa}$ for $\alpha < \boldsymbol{h}_{\mathfrak{I}}$ such that the following hold:*

(i) *$Q_\alpha \leqslant Q_\beta$ for all $\beta < \alpha$;*

(ii) *$|\{D \in Q_{\alpha+1} : D \cap B \in \mathfrak{I}^+\}| \geqslant \rho$ for every $B \in Q_\alpha$;*

(iii) *given $C \in \mathfrak{I}^+$, there is an $\alpha < \boldsymbol{h}_{\mathfrak{I}}$ such that for every $B \in Q_\alpha$, $C - B \in \mathfrak{I}^+$;*

(iv) *$Q_{\alpha+1} \cap P(A) \neq 0$ whenever $A \in \mathfrak{I}^+$ and $\alpha < \boldsymbol{h}_{\mathfrak{I}}$ are such that $A \in \Theta_{Q_\alpha}$;*

(v) *$[B]_{\mathfrak{I}} \in Z$ for every $B \in \bigcup_{\alpha < \boldsymbol{h}_{\mathfrak{I}}} Q_\alpha$;*

(vi) *Suppose that $\rho = 2^\kappa$. Let $A \in \mathfrak{I}^+$ and $\alpha < \boldsymbol{h}_{\mathfrak{I}}$ be such that $A \in \Theta_{Q_\alpha}$, and let $F : [A]^2 \to 2$ be such that for every $D \in \mathfrak{I}^+ \cap P(A)$, there is an $E \in \mathfrak{I}^+ \cap P(D)$ with F being constant on $[E]^2$. Then there is a $B \in Q_{\alpha+1} \cap P(A)$ with F being constant on $[B]^2$.*

Proof. Similar to the proof of Proposition 13.1 in [23]. □

Proposition 2.15. *Assume that $\mathfrak{I}$ is nowhere $\boldsymbol{h}_{\mathfrak{I}}$-distributive, $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$ has a dense κ-closed subset and $2^{<\kappa} < \boldsymbol{h}_{\mathfrak{I}}$.* Let *$Q_\alpha \in M_{\mathfrak{I},\kappa}$ for $\alpha < \boldsymbol{h}_{\mathfrak{I}}$ be such that $Q_\alpha \leqslant Q_\beta$ whenever $\beta < \alpha$, and for every $C \in \mathfrak{I}^+$, there is an $\alpha < \boldsymbol{h}_{\mathfrak{I}}$ with $\{C - B : B \in Q_\alpha\} \subseteq \mathfrak{I}^+$. Then $\mathfrak{I}^+ = \bigcup_{\alpha < \boldsymbol{h}_{\mathfrak{I}}} \Theta_{Q_\alpha}$.*

Proof. A straightforward modification of the proof of Proposition 13.1 in [23]. □

Assume that $\mathfrak{I}$ is nowhere $\boldsymbol{h}_{\mathfrak{I}}$-distributive. Fix $Z \subseteq \mathfrak{I}^+/\mathfrak{I}$ such that Z is dense in $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$. Select a cardinal ρ so that $\rho = 2^\kappa$ in case $sat(\mathfrak{I} \mid A) = (2^\kappa)^+$ for every $A \in \mathfrak{I}^+$, and $2 \leqslant \rho < \bigcap_{A \in \mathfrak{I}^+} sat(\mathfrak{I} \mid A)$ otherwise. Then let $Q_\alpha \in M_{\mathfrak{I},\kappa}$ for $\alpha < \boldsymbol{h}_{\mathfrak{I}}$ satisfy the conditions (i)–(vi) of Proposition 2.14. Let μ be a regular infinite cardinal with $\mu \leqslant \boldsymbol{h}_{\mathfrak{I}}$. Let N_μ be the set of all $k \in \bigcup_{\gamma < \mu} \prod_{\alpha < \gamma} Q_\alpha$ such that $k(\alpha) - k(\beta) \in \mathfrak{I}$ whenever $\beta < \alpha$. Then define $\mathfrak{I}_\mu \subseteq P(\kappa)$ by letting $B \in \mathfrak{I}_\mu$ if and only if $\{k \in N_\mu : \exists \alpha \in dom(k)(B \cap k(\alpha) \in \mathfrak{I})\}$ is dense in $(N_\mu, \subseteq)$. $\mathfrak{I}_\mu$ is easily seen to be an ideal over κ with $\mathfrak{I} \subseteq \mathfrak{I}_\mu \subset P(\kappa)$. We have $Q_\alpha \in M_{\mathfrak{I}_\mu,\kappa}$ for every $\alpha < \mu$. For every $C \in \mathfrak{I}_\mu^+$ and every $h \in \prod_{\alpha < \mu} Q_\alpha$, $\{C - h(\alpha) : \alpha < \mu\} \cap \mathfrak{I}_\mu^+ \neq 0$. Hence by Proposition 2.5, $\mathfrak{I}_\mu$ is nowhere μ-distributive.

Let us now assume that $sat(\mathfrak{I} \mid A) = (2^\kappa)^+$ for every $A \in \mathfrak{I}^+$. Then clearly, $\mathfrak{I}_\mu^+ = \{C \subseteq \kappa : \exists B \in \bigcup_{\alpha < \mu} Q_\alpha (B \subseteq C)\}$. Hence, if Z is ν-closed for every infinite cardinal $\nu < \mu$, then $(\mathfrak{I}_\mu^+/\mathfrak{I}_\mu, \leqslant)$ admits a dense subset that is ν-closed for every infinite cardinal $\nu < \mu$, and therefore $\mu = \boldsymbol{h}_{\mathfrak{I}_\mu}$ ($=\boldsymbol{k}_{\mathfrak{I}_\mu}$ in case $\kappa = \omega$ and $\mathfrak{I} = [\omega]^{<\omega}$). If $\mathfrak{I}^+ \to (\mathfrak{I}^+)^2$, then clearly $\mathfrak{I}_\mu^+ \to (\mathfrak{I}_\mu^+)^2$.

We will finally show that if there is over some $\mu \leqslant \kappa$ a (nowhere prime) ideal K that satisfies $\boldsymbol{h}_K > \kappa$, then there exists a (nowhere prime) κ-distributive ideal J over κ.

For the remainder of the section we let μ, K and A_α and J_α for $\alpha < \mu$ be as follows:

(0) μ is a regular infinite cardinal $\leqslant \kappa$;

(1) K is a nowhere prime ideal over μ such that $\mu \subseteq K \subset P(\mu)$;
(2) $A_\alpha \in [\kappa]^\kappa$, and $A_\alpha \cap A_\beta = 0$ for all $\beta < \mu$ with $\alpha \neq \beta$;
(3) J_α is a prime ideal over A_α such that $[A_\alpha]^{<\kappa} \subseteq J_\alpha \subset P(A_\alpha)$.

We define $J \subseteq P(\kappa)$ by letting $E \in J$ if and only if $\{\alpha < \mu\colon E \cap A_\alpha \in J_\alpha^+\} \in K$.

It is immediate that J is an ideal over κ that satisfies $\kappa \subseteq J \subset P(\kappa)$. Moreover, J is tall and nowhere prime and $add(J) \leqslant add(K)$. Notice that if the set $\{\alpha < \mu\colon add(J_\alpha) = \lambda\}$ belongs to K^+ (respectively to K^*), then $add(J)$ is less than or equal to (resp. is equal to) $\lambda \cap add(K)$.

Proposition 2.16. (i) $\boldsymbol{h}_J = \boldsymbol{h}_K$.

(ii) *Given an infinite cardinal ρ, $(J^+/J, \leqslant)$ is ρ-closed if and only if $(K^+/K, \leqslant)$ is ρ-closed.*

Proof. We will prove (i) and leave the (similar) proof of (ii) to the reader. We start by defining $\varphi\colon J^+ \to K^+$ and $\psi\colon K^+ \to J^+$ by letting $\varphi(T) = \bigcup_{\alpha \in T} A_\alpha$ and $\psi(E) = \{\alpha < \mu\colon E \cap A_\alpha \in J_\alpha^+\}$. Let us now show that $\boldsymbol{h}_J \leqslant \boldsymbol{h}_K$. Thus, let $Q_\beta \in M_{K,\mu}$ for $\beta < \boldsymbol{h}_K$ and $Z \in K^+$ be such that for every $S \in K^+ \cap P(Z)$, there is a $\beta < \boldsymbol{h}_K$ with $|\{T \in Q_\beta\colon S \cap T \in K^+\}| \geqslant 2$. For each $\beta < \boldsymbol{h}_K$, set $R_\beta = \{\varphi(E)\colon E \in Q_\beta\}$. Clearly, $R_\beta \in M_{J,\kappa}$. Now fix $C \in J^+ \cap P(\varphi(Z))$. We have $\psi(C) \subseteq Z$. Hence there are $\beta < \boldsymbol{h}_K$ and $T_0, T_1 \in Q_\beta$ such that for each $i < 2$, $\psi(C) \cap T_i \in K^+$. Clearly, $C \cap \varphi(T_i) \in J^+$ for each $i < 2$. Thus, $|\{D \in R_\beta\colon C \cap D \in J^+\}| \geqslant 2$.

We will next show that $\boldsymbol{h}_J \geqslant \boldsymbol{h}_K$. Thus, let ν be a cardinal with $0 < \nu < \boldsymbol{h}_K$, and let $R_\beta \in M_{J,\kappa}$ for $\beta < \nu$. For each $\beta < \nu$, set $Q_\beta = \{\psi(D)\colon D \in R_\beta\}$. It is readily checked that $Q_\beta \in M_{K,\mu}$. Pick $Q \in M_{K,\mu}$ so that for every $\beta < \nu$, $Q \leqslant Q_\beta$. Now set $R = \{\varphi(E)\colon E \in Q\}$. Given $E \in Q$ and $\beta < \nu$, there is a $D \in R_\beta$ with $E - \psi(D) \in K$. Then $\varphi(E) - D \in J$, as $\varphi(E) - D \subseteq \bigcup_{\alpha \in E - \psi(D)} A_\alpha \cup \bigcup_{\alpha \in E \cap \psi(D)} (A_\alpha - D)$. Thus, $R \leqslant R_\beta$ for all $\beta < \nu$. □

Thus, assuming $\kappa < \boldsymbol{h}_{[\omega]^{<\omega}}$, there exists a κ-distributive nowhere prime ideal J over κ with $\kappa \subseteq J \subset P(\kappa)$. The existence of such ideals over larger (successor) cardinals can be obtained by combining Proposition 2.16 with the following result of Moti Gitik [11]. Assume that μ is $P^{n+3}(\mu)$-hypermeasurable, where $n \in \omega$. Then there is a generic extension of the universe where μ is no longer measurable and there is a weakly selective (in fact normal) ideal K over μ such that $\mu \subseteq K \subset P(\mu)$ and $(K^+/K, \leqslant)$ is $\mu^{+(n+1)}$-closed (and $\boldsymbol{h}_K = \mu^{+(n+2)}$). Gitik [11] also constructed models where μ is measurable (or even supercompact) and there is a K as above.

3. Almost κ-distributive ideals

We will follow Kastanas [17] and base our generalization to Ellentuck's theorem [9] on two-person games. The approach will consist in eliminating any reference to the second player (see e.g. the definition of N_3 in Section 5 and that of C_3 in Section 6;

both are expressed solely in terms of player I). We now introduce a property (Proposition 3.1(i)) that will guarantee that this can be done (see Proposition 7.1). Proposition 3.1 shows that the property can be seen as a weak form of κ-distributivity. In case $add(\mathfrak{J}) = \kappa$ the weakening is actually only apparent, as is shown by Proposition 3.3.

$\mathfrak{J}$ is *almost κ-distributive* if given $Q_\alpha \in M_{\mathfrak{J},\kappa}$ for $\alpha < \kappa$, there exists $Q \in M_{\mathfrak{J},\kappa}$ with the following property: for every $A \in Q$ and every $\alpha \in A$, there is a $B \in Q_\alpha$ with $A - B \in \mathfrak{J}$.

Proposition 3.1. *The following are equivalent:*

(i) *Let $C \in \mathfrak{J}^+$, and let $\varphi : \mathfrak{J}^+ \cap P(C) \to \kappa$ and $\psi : \mathfrak{J}^+ \cap P(C) \to \mathfrak{J}^+$ be such that $\varphi(A) \in A$ and $\psi(A) \subseteq A$. Then there exists $E \in \mathfrak{J}^+ \cap P(C)$ such that for every $\alpha \in E$, there is an $A \in \mathfrak{J}^+ \cap P(C)$ with $\varphi(A) = \alpha$ and $E - \psi(A) \in \mathfrak{J}$.*

(ii) *$\mathfrak{J}$ is almost κ-distributive.*

(iii) *Let $C \in \mathfrak{J}^+$, and let $Q_\alpha \in M_{\mathfrak{J},C}$ for $\alpha \in C$. Then there exists $Q \in M_{\mathfrak{J},C}$ with the following property: for every $A \in Q$ and every $\alpha \in A$, there is a $B \in Q_\alpha$ with $A - B \in \mathfrak{J}$.*

Proof. (i) $\to$ (ii): Assume (i), and let $Q_\alpha \in M_{\mathfrak{J},\kappa}$ for $\alpha < \kappa$. Fix $C \in \mathfrak{J}^+$. Define $\varphi : \mathfrak{J}^+ \cap P(C) \to \kappa$ and $\psi : \mathfrak{J}^+ \cap P(C) \to \mathfrak{J}^+$ as follows. Put $\varphi(A) = \bigcap A$. Choose $B \in Q_{\bigcap A}$ with $B \cap A \in \mathfrak{J}^+$, and set $\psi(A) = B$. Let $E \in \mathfrak{J}^+ \cap P(C)$ be such that for every $\alpha \in E$, there is an $A \in \mathfrak{J}^+ \cap P(C)$ with $\varphi(A) = \alpha$ and $E - \psi(A) \in \mathfrak{J}$. Clearly, for every $\alpha \in E$, there is a $B \in Q_\alpha$ with $E - B \in \mathfrak{J}$. It is now easy to see that (ii) holds.

(ii) $\to$ (iii): Assume (ii), and fix $C \in \mathfrak{J}^+$ and $Q_\alpha \in M_{\mathfrak{J},C}$ for $\alpha \in C$. Choose $R_\alpha \in M_{\mathfrak{J},\kappa}$ for $\alpha \in C$ with $Q_\alpha \subseteq R_\alpha$. Set $R_\alpha = \{\kappa\}$ for each $\alpha \in \kappa - C$. Select $R \in M_{\mathfrak{J},\kappa}$ so that for every $A \in R$ and every $\alpha \in A$, there is a $B \in R_\alpha$ with $A - B \in \mathfrak{J}$. Now put $Q = \mathfrak{J}^+ \cap \{A \cap C\colon A \in R\}$. Then $Q \in M_{\mathfrak{J},C}$. Moreover for every $A \in Q$ and every $\alpha \in A$, there is a $B \in Q_\alpha$ with $A - B \in \mathfrak{J}$.

(iii) $\to$ (i): Assume (iii). Let $C \in \mathfrak{J}^+$, and let $\varphi : \mathfrak{J}^+ \cap P(C) \to \kappa$ and $\psi : \mathfrak{J}^+ \cap P(C) \to \mathfrak{J}^+$ be such that $\varphi(A) \in A$ and $\psi(A) \subseteq A$. Given $\gamma \in C$, let S_γ be the set of all $Q \in D_{\mathfrak{J},C}$ such that for every $B \in Q$, there exists $A \in \mathfrak{J}^+ \cap P(C)$ with $\varphi(A) = \gamma$ and $B \subseteq \psi(A)$. Then let Q_γ be a maximal element of $(S_\gamma, \subseteq)$, and select $R_\gamma \in M_{\mathfrak{J},C}$ with $Q_\gamma \subseteq R_\gamma$. Select $Q \in M_{\mathfrak{J},C}$ such that for every $B \in Q$ and every $\gamma \in B$, there is a $D \in R_\gamma$ with $B - D \in \mathfrak{J}$. Choose $B \in Q$, and let $h \in \prod_{\gamma \in B} R_\gamma$ be such that $B - h(\gamma) \in \mathfrak{J}$. Set $E = \{\alpha \in B\colon h(\alpha) \in Q_\alpha\}$. We claim that $B - E \in \mathfrak{J}$. Suppose otherwise. Set $\gamma = \varphi(B - E)$. Then $\gamma \in B$ and $h(\gamma) \in R_\gamma - Q_\gamma$. Moreover, $\psi(B - E) - h(\gamma) \in \mathfrak{J}$. It follows that $Q_\gamma \cup \{\psi(B - E) \cap h(\gamma)\} \in S_\gamma$, a contradiction. Hence, $E \in \mathfrak{J}^+$. Given $\alpha \in E$, there is an $A \in \mathfrak{J}^+ \cap P(C)$ with $\varphi(A) = \alpha$ and $h(\alpha) \subseteq \psi(A)$. Clearly, $E - \psi(A) \in \mathfrak{J}$. $\square$

The remainder of the section is devoted to the problem of determining how much distributivity is implied by almost κ-distributivity.

Proposition 3.2. *The following are equivalent:*

(i) *$\mathfrak{J}$ is μ-distributive for every infinite cardinal $\mu < \kappa$, and almost κ-distributive.*

(ii) *$\mathfrak{J}$ is κ-distributive.*

Proof. (i) $\to$ (ii): Let $Q_\alpha \in M_{\mathfrak{I},\kappa}$ for $\alpha < \kappa$. Select $R_\alpha \in M_{\mathfrak{I},\kappa}$ for $\alpha < \kappa$ so that $R_\alpha \leqslant Q_\beta$ for all $\alpha, \beta \in \kappa$ with $\beta \leqslant \alpha$. Pick $Q \in M_{\mathfrak{I},\kappa}$ such that for every $A \in Q$ and every $\alpha \in A$, there is a $B \in R_\alpha$ with $A - B \in \mathfrak{I}$. Clearly $Q \leqslant Q_\alpha$ for every $\alpha < \kappa$.

(ii) $\to$ (i): Trivial. □

Proposition 3.3. *Assume that $\mathfrak{I}$ is almost κ-distributive. Then $\boldsymbol{h}_{\mathfrak{I}} > add(\mathfrak{I})$.*

Proof. Let $Q_\alpha \in M_{\mathfrak{I},\kappa}$ for $\alpha < add(\mathfrak{I})$. For every $A \in \mathfrak{I}^+$, set $Z_A = \{\alpha < add(\mathfrak{I})$: $\forall B \in Q_\alpha(A - B \in \mathfrak{I}^+)\}$. Let us assume that there exists $C \in \mathfrak{I}^+$ such that $Z_A \neq 0$ for all $A \in \mathfrak{I}^+ \cap P(C)$. Define $\varphi : \mathfrak{I}^+ \cap P(C) \to \kappa$, $\psi : \mathfrak{I}^+ \cap P(C) \to \mathfrak{I}^+$ and $\chi : \mathfrak{I}^+ \cap P(C) \to add(\mathfrak{I})$ so that

(i) $\chi(A) = \bigcap Z_A$;
(ii) $\psi(A) \in Q_{\chi(A)}$;
(iii) $A \cap \psi(A) \in \mathfrak{I}^+$;
(iv) $\varphi(A) \in A - \psi(A)$.

Select $E \in \mathfrak{I}^+ \cap P(C)$ and $\eta : E \to \mathfrak{I}^+ \cap P(C)$ so that for every $\beta \in E$, $\varphi(\eta(\beta)) = \beta$ and $E - \psi(\eta(\beta)) \in \mathfrak{I}$. Clearly, for every $\alpha \in \bigcup_{\beta \in E} \chi(\eta(\beta))$, there is a $B \in Q_\alpha$ with $E - B \in \mathfrak{I}$. As $Z_E \neq 0$, we have $\bigcup_{\beta \in E} \chi(\eta(\beta)) < add(\mathfrak{I})$. Hence there are $H \in \mathfrak{I}^+ \cap P(E)$ and $\alpha \in \bigcup_{\beta \in E} \chi(\eta(\beta))$ such that $\chi(\eta(\beta)) = \alpha$ for every $\beta \in H$. Given $\beta \in H$, we have $H \cap \psi(\eta(\beta)) = 0$ and $E - \psi(\eta(\beta)) \in \mathfrak{I}$, a contradiction. □

Corollary 3.4. *Assuming $\kappa \in \{\omega, \omega_1\}$, $\mathfrak{I}$ is κ-distributive if and only if $\mathfrak{I}$ is almost κ-distributive.*

Proof. By Propositions 3.2 and 3.3. □

4. $G_{\mathfrak{I}}(a, C, A)$

We now introduce the games that will be the main tool in our approach to the combinatorial theory of ideals.

For each $\alpha \in \omega + 1$, we set $\Xi_\alpha = (\mathfrak{I}^+)^\alpha \times \kappa^\alpha \times (\mathfrak{I}^+)^\alpha$.

Let $\alpha \in (\omega + 1) - 1$, and let $a \in [\kappa]^{<\omega}$, $C \in \mathfrak{I}^+$ and $A \subseteq \Xi_\alpha$ be given. We define the two-person game $G_{\mathfrak{I}}(a, C, A)$ as follows:

Each player makes α moves. I starts by selecting $A_0 \in \mathfrak{I}^+ \cap P(C)$; II then chooses $\alpha_0 \in \{\beta \in A_0 : a \subseteq \beta\}$ and $B_0 \in \mathfrak{I}^+ \cap P(A_0 - (\alpha_0 + 1))$; I now picks $A_1 \in \mathfrak{I}^+ \cap P(B_0)$; II answers by playing $\alpha_1 \in A_1$ and $B_1 \in \mathfrak{I}^+ \cap P(A_1 - (\alpha_1 + 1))$; etc.

I is said to win if $(f, g, h) \in A$, where for every $i < \alpha$, $f(i) = A_i$, $g(i) = \alpha_i$ and $h(i) = B_i$.

Given $W \subseteq [\kappa]^{|a| + \alpha}$, we let $G_{\mathfrak{I}}(a, C, W)$ stand for $G_{\mathfrak{I}}(a, C, \{(f, g, h) \in \Xi_\alpha$: $a \cup ran(g) \in W\})$.

In this section we establish some basic facts concerning these games. The ideal we are working with will not be assumed to satisfy any special property (with the exception of Proposition 4.5). We first observe that it is easy to 'switch' from player I to player II. We will later (see Proposition 7.1) use distributivity to move in the other direction, that is from player II to player I.

Proposition 4.1. *Let $\alpha \in (\omega + 1) - 1$, $a \in [\kappa]^{<\omega}$, $W \subseteq [\kappa]^{|a|+\alpha}$ and $C \in \mathfrak{I}^+$. Assume that* I *has a winning strategy in $G_{\mathfrak{I}}(a, C, W)$. Then* II *has a winning strategy in $G_{\mathfrak{I}}(a, C', [\kappa]^{|a|+\alpha} - W)$ for some $C' \in \mathfrak{I}^+ \cap P(C)$.*

Proof. Let σ be a winning strategy for I in $G_{\mathfrak{I}}(a, C, W)$. We will define a winning strategy τ for II in $G_{\mathfrak{I}}(a, \sigma(0), [\kappa]^{|a|+\alpha} - W)$. Let I's successive moves be A_i for $i < \alpha$. For each $i < \alpha$, put $\beta_i = \bigcap \{\beta \in A_i\colon a \subseteq \beta\}$ and $B_i = A_i - (\beta_i + 1)$. Now set

(0) $\tau(A_0, \ldots, A_j) = (\beta_j, \sigma((\beta_0, B_0), \ldots, (\beta_j, B_j)))$ for every $j \in \bigcup \alpha$;

(1) $\tau(A_0, \ldots, A_{\bigcup \alpha}) = (\beta_{\bigcup \alpha}, B_{\bigcup \alpha})$ in case $\alpha < \omega$. □

Corollary 4.2. *Let $\alpha \in (\omega + 1) - 1$, $a \in [\kappa]^{<\omega}$, $W \subseteq [\kappa]^{|a|+\alpha}$ and $C \in \mathfrak{I}^+$. Assume that for every $A \in \mathfrak{I}^+ \cap P(C)$,* I *has a winning strategy in $G_{\mathfrak{I}}(a, A, W)$. Then* II *has a winning strategy in $G_{\mathfrak{I}}(a, C, [\kappa]^{|a|+\alpha} - W)$.*

Proof. Using Proposition 4.1, define E_A and τ_A for $A \in \mathfrak{I}^+ \cap P(C)$ so that $E_A \in \mathfrak{I}^+ \cap P(A)$ and τ_A is a winning strategy for II in $G_{\mathfrak{I}}(a, E_A, [\kappa]^{|a|+\alpha} - W)$. We define a winning strategy τ for II in $G_{\mathfrak{I}}(a, C, [\kappa]^{|a|+\alpha} - W)$ by letting $\tau(A_0, A_1, \ldots, A_i) = \tau_{A_0}(E_{A_0}, A_1, \ldots, A_i)$. □

We let $\Sigma_{\mathfrak{I}}^{\alpha}$ be the set of all strategies for I in $G_{\mathfrak{I}}(0, C, \Xi_\alpha)$.

Let $\sigma \in \Sigma_{\mathfrak{I}}^{\alpha}$ and $a \in [\kappa]^{<\omega}$. Given $\delta \in \alpha + 1$, we let $P_{a,\sigma}^{\delta}$ be the set of all $(g, h) \in \kappa^{\delta} \times (\mathfrak{I}^+)^{\delta}$ such that the following holds: $g(0) \in \{\beta \in \sigma(0)\colon\ a \subseteq \beta\}$ and $h(0) \subseteq \sigma(0) - (g(0) + 1)$; $g(1) \in \sigma(g(0), h(0))$ and $h(1) \subseteq \sigma(g(0), h(0)) - (g(1) + 1)$; etc.

We define $\langle a, \sigma \rangle \subseteq [\kappa]^{|a|+\alpha}$ by letting $D \in \langle a, \sigma \rangle$ if and only if there exists $(g, h) \in P_{a,\sigma}^{\alpha}$ with $D = a \cup ran(g)$.

We set $P_{\mathfrak{I}} = \{\langle a, \sigma \rangle\colon a \in [\kappa]^{<\omega}$ *and* $\sigma \in \Sigma_{\mathfrak{I}}^{\omega}\}$.

Given $\sigma, \sigma' \in \Sigma_{\mathfrak{I}}^{\omega}$, we let $\sigma' \leqslant \sigma$ just in case $\sigma'(0) \subseteq \sigma(0)$, $\sigma'(\alpha_0, B_0) \subseteq \sigma(\alpha_0, B_0)$, etc.

It is immediate that $\leqslant$ partially orders $\Sigma_{\mathfrak{I}}^{\omega}$.

For each $\sigma \in \Sigma_{\mathfrak{I}}^{\omega}$, we set $\Sigma_{\mathfrak{I}}^{\omega} | \sigma = \{\sigma' \in \Sigma_{\mathfrak{I}}^{\omega}\colon \sigma' \leqslant \sigma\}$.

Given $A \in \mathfrak{I}^+$, we define $\ulcorner A \urcorner \in \Sigma_{\mathfrak{I}}^{\omega}$ by letting $\ulcorner A \urcorner(0) = A$ and $\ulcorner A \urcorner((\alpha_0, B_0), \ldots, (\alpha_i, B_i)) = B_i$.

We observe that $\Sigma_{\mathfrak{I}}^{\omega} | \ulcorner A \urcorner = \{\sigma \in \Sigma_{\mathfrak{I}}^{\omega}\colon \sigma(0) \subseteq A\}$.

The next proposition testifies to the Ramseyian character of our games, since it asserts the existence of a structured family of sets such that every subset of a fixed order type of any one of them satisfies the same given property.

Proposition 4.3. *There is an u: $\bigcup_{\alpha \in (\omega+1)-1} \Sigma_{\mathfrak{I}}^{\alpha} \to \Sigma_{\mathfrak{I}}^{\omega}$ with the following properties:* (i) *$[D]^{\alpha} \subseteq \langle 0, \sigma \rangle$ whenever $\sigma \in \Sigma_{\mathfrak{I}}^{\alpha}$ and $D \in \langle 0, u(\sigma) \rangle$;* (ii) *$u(\sigma) \leqslant \sigma$ for all $\sigma \in \Sigma_{\mathfrak{I}}^{\omega}$.*

Proof. Let $\alpha \in (\omega + 1) - 1$ and $\sigma \in \Sigma_{\mathfrak{I}}^{\alpha}$. We define $u(\sigma)$ as follows. We put $u(\sigma) = \ulcorner \sigma(0) \urcorner$ in case $\alpha = 1$. Let us now assume that $\alpha > 1$. Let II successively play

(β_i, B_i) for $i < \omega$. Put $D = \{\beta_i : i < \omega\}$. Given $a, b \in [D]^{<\alpha}$, we set $a < b$ just in case one of the following four conditions is satisfied:

(i) $a = 0$ and $b \neq 0$.

(ii) $a, b \neq 0$ and $\bigcup a < \bigcup b$.

(iii) $a, b \neq 0$, $\bigcup a = \bigcup b$ and $|a| > |b|$.

(iv) $a, b \neq 0$, $\bigcup a = \bigcup b$, $|a| = |b|$ and $a - \{\bigcup a\} < b - \{\bigcup b\}$.

It is easy to check that $<$ linearly orders $[D]^{<\alpha}$. For each $b \in [D]^{<\alpha} - \{0\}$, we let b^- denote the greatest element of the set $\{a \in [D]^{<\alpha}: a < b\}$. We now define A_a for $a \in [D]^{<\alpha}$ and C_b for $b \in [D]^{<\alpha} - \{0\}$ as follows:

(0) Assume that $b^- = 0$, or that $\bigcup b^- < \bigcup b$, and let $\bigcup b = \beta_k$. Then $C_b = B_k$;

(1) $C_b = A_{b^-}$ whenever $b^- \neq 0$ and $\bigcup b^- = \bigcup b$;

(2) $A_0 = \sigma(0)$;

(3) $A_{\{\beta_{i_0}, \ldots, \beta_{i_k}\}} = \sigma((\beta_{i_0}, C_{\{\beta_{i_0}\}}), (\beta_{i_1}, C_{\{\beta_{i_0}, \beta_{i_1}\}}), \ldots, (\beta_{i_k}, C_{\{\beta_{i_0}, \ldots, \beta_{i_k}\}}))$ whenever $i_0 \in i_1 \in \ldots \in i_k \in \alpha$.

Observe that $A_b \subseteq A_a$ whenever $a, b \in [D]^{<\alpha}$ are such that $a < b$. Finally, set $u(\sigma)(0) = A_0$, and $u(\sigma)((\beta_0, B_0), \ldots, (\beta_i, B_i)) = A_{\{\beta_i\}}$. $\square$

Given $\sigma \in \Sigma_{\mathfrak{I}}^{\omega}$, we let $\Phi(\sigma)$ be the set of all $\sigma' \in \Sigma_{\mathfrak{I}}^{\omega} | \ulcorner \sigma(0) \urcorner$ with the following property: there is a $\rho : \bigcup_{i < \omega} P_{0,\sigma'}^{i+1} \to \mathfrak{I}^+$ such that for every $i < \omega$ and every $(g, h) \in P_{0,\sigma'}^{i+1}$,

$$\sigma'((g(0), h(0)), \ldots, (g(i), h(i))) \subseteq \sigma((g(0), \rho(g|1, h|1)), \ldots, (g(i), \rho(g, h))).$$

We let $\Psi(\sigma)$ be the set of all $\sigma' \in \Sigma_{\mathfrak{I}}^{\omega} | \ulcorner \sigma(0) \urcorner$ such that there is a $\rho : \bigcup_{i < \omega} P_{0,\sigma'}^{i+1} \to \mathfrak{I}^+$ with the following property: for every $i < \omega$ and every $(g, h) \in P_{0,\sigma'}^{i+1}$, $\rho(g, h) \subseteq h(i)$ and

$$\sigma'((g(0), h(0)), \ldots, (g(i), h(i))) \subseteq \sigma((g(0), \rho(g|1, h|1)), \ldots, (g(i), \rho(g, h)))).$$

Clearly, $\Sigma_{\mathfrak{I}}^{\omega} | \sigma \subseteq \Psi(\sigma) \subseteq \Phi(\sigma)$. Notice that $\langle a, \sigma' \rangle \subseteq \langle a, \sigma \rangle$ whenever $\sigma' \in \Phi(\sigma)$ and $a \in [\kappa]^{<\omega}$.

Proposition 4.4. *Let $\sigma, \sigma' \in \Sigma_{\mathfrak{I}}^{\omega}$ be such that $\sigma(0) \cap \sigma'(0) \in \mathfrak{I}^+$. Then $\Sigma_{\mathfrak{I}}^{\omega} | \sigma \cap \Psi(\sigma') \neq 0$.*

Proof. Define $\sigma'' \in \Sigma_{\mathfrak{I}}^{\omega}$ as follows: $\sigma''(0) = \sigma(0) \cap \sigma'(0)$; $\sigma''(\alpha_0, B_0) = \sigma'(\alpha_0, \sigma(\alpha_0, B_0))$;

$$\sigma''((\alpha_0, B_0), (\alpha_1, B_1)) = \sigma'((\alpha_0, \sigma(\alpha_0, \sigma(\alpha_0, B_0)), (\alpha_1, \sigma((\alpha_0, B_0), (\alpha_1, B_1))));$$

etc. Clearly, $\sigma'' \in \Sigma_{\mathfrak{I}}^{\omega} | \sigma \cap \Psi(\sigma')$. $\square$

Proposition 4.5. *Let ν be a cardinal such that $add(\mathfrak{I}) > \nu > 0$ and $\mathfrak{I}$ is ν-distributive. Furthermore, let $\sigma \in \Sigma_{\mathfrak{I}}^{\omega}$, and let $S_\gamma \subseteq \Sigma_{\mathfrak{I}}^{\omega}$ for $\gamma < \nu$ be such that for every $A \in \mathfrak{I}^+ \cap P(\sigma(0))$, $S_\gamma \cap \Sigma_{\mathfrak{I}}^{\omega} | \ulcorner A \urcorner \neq 0$. Then $\Sigma_{\mathfrak{I}}^{\omega} | \sigma \cap (\bigcap_{\gamma < \nu} \bigcup_{\tau \in S_\gamma} \Psi(\tau)) \neq 0$.*

Proof. We define $\sigma' \in \Sigma_{\mathfrak{I}}^{\omega} | \sigma$ as follows. For each $\gamma < \nu$, set $K_\gamma = \{Q \in D_{\mathfrak{I}, \sigma(0)}: Q \subseteq \{\tau(0): \tau \in S_\gamma\}\}$, and let Q_γ be a maximal element of $(K_\gamma, \subseteq)$. It is easy to verify that each $Q_\gamma \in M_{\mathfrak{I}, \sigma(0)}$. Select $k \in \prod_{\gamma < \nu} Q_\gamma$ with $\bigcap_{\gamma < \nu} k(\gamma) \in \mathfrak{I}^+$, and set $\sigma'(0) = \bigcap_{\gamma < \nu} k(\gamma)$. Pick

$\sigma_\gamma \in S_\gamma$ for $\gamma < \nu$ with $\sigma_\gamma(0) = k(\gamma)$. Now assume II successively plays (α_i, B_i) for $i < \omega$. For each i, put $H_i = \sigma((\alpha_0, B_0), \ldots, (\alpha_i, B_i))$. By induction on i, we define E_γ^i for $\gamma < \nu$ and g_i as follows. Let L_γ^i be the set of all $Q \in D_{\mathfrak{J}, H_i}$ such that

$$Q \subseteq \{\sigma_\gamma((\alpha_0, E_\gamma^0), (\alpha_1, E_\gamma^1), \ldots, (\alpha_i, B)) \colon B \in \mathfrak{J}^+ \cap P(H_i)\},$$

and let Q_γ^i be a maximal element of $(L_\gamma^i, \subseteq)$. We easily have $Q_\gamma^i \in M_{\mathfrak{J}, H_i}$. Select $g_i \in \prod_{\gamma < \nu} Q_\gamma^i$ with $\bigcap_{\gamma < \nu} g_i(\gamma) \in \mathfrak{J}^+$, and choose $E_\gamma^i \in \mathfrak{J}^+ \cap P(H_i)$ for $\gamma < \nu$ so that $g_i(\gamma) = \sigma_\gamma((\alpha_0, E_\gamma^0), (\alpha_1, E_\gamma^1), \ldots, (\alpha_i, E_\gamma^i))$. We put $\sigma'((\alpha_0, B_0), \ldots, (\alpha_i, B_i)) = \bigcap_{\gamma < \nu} g_i(\gamma)$. Clearly, $\sigma' \in \Sigma_{\mathfrak{J}}^{\omega} \mid \sigma \cap \bigcap_{\gamma < \nu} \Psi(\sigma_\gamma)$. □

Lemma 4.6. *Let $\Lambda_i \subseteq \Xi_i$ for $i < \omega$, and let*

$$\Lambda = \{(f, g, h) \in \Xi_\omega \colon \exists i < \omega((f \mid i, g \mid i, h \mid i) \in \Lambda_i)\}.$$

Then $G_{\mathfrak{J}}(a, C, \Lambda)$ is determined for all $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{J}^+$.

Proof. Fix $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{J}^+$. Assume I has no winning strategy in $G_{\mathfrak{J}}(a, C, \Lambda)$. We will define a winning strategy τ for II in $G_{\mathfrak{J}}(a, C, \Lambda)$. Suppose I's successive moves are A_i for $i < \omega$. We choose α_i and B_i for $i < \omega$ so that for every i, I has no winning strategy in $G_{\mathfrak{J}}(a \cup \{\alpha_j \colon j \leqslant i\}, B_i, \Gamma_i)$, where Γ_i is defined as follows: $(f, g, h) \in \Gamma_i$ if and only if $(f', g', h') \in \Lambda$, where for each $j \leqslant i$, $f'(j) = A_j$, $g'(j) = \alpha_j$ and $h'(j) = B_j$, and for each $r > i$, $f'(r) = f(r - i - 1)$, $g'(r) = g(r - i - 1)$ and $h'(r) = h(r - i - 1)$. Then put $\tau(A_0) = (\alpha_0, B_0), \tau(A_0, A_1) = (\alpha_1, B_1)$, etc. □

5. $N_{\mathfrak{J}}$

The collection of nowhere Ramsey sets consists of all those subsets of $[\omega]^\omega$ that are nowhere dense in the Ellentuck topology. To generalize the notion to arbitrary ideals we will rely on games rather than on topology. By a result of [17], our definition is equivalent to the one given above in case $\mathfrak{J} = [\omega]^{<\omega}$.

Let $N_{\mathfrak{J}}$ be the set of all $W \subseteq [\kappa]^\omega$ such that for every $a \in [\kappa]^{<\omega}$ and every $C \in \mathfrak{J}^+$, I has a winning strategy in $G_{\mathfrak{J}}(a, C, [\kappa]^\omega - W)$.

Proposition 5.1. *Given $W \subseteq [\kappa]^\omega$, the following are equivalent:*
(i) *$W \in N_{\mathfrak{J}}$.*
(ii) *For every $\langle a, \sigma \rangle \in P_{\mathfrak{J}}$, there is a $\sigma' \in \Sigma_{\mathfrak{J}}^{\omega} \mid \sigma$ with $\langle a, \sigma' \rangle \cap W = 0$.*

Proof. (i) → (ii): Let $a \in [\kappa]^{<\omega}$ and $\sigma \in \Sigma_{\mathfrak{J}}^{\omega}$ be given. Then I has a winning strategy σ' in $G_{\mathfrak{J}}(a, \sigma(0), [\kappa]^\omega - W)$. By Proposition 4.4, there is a $\sigma'' \in \Sigma_{\mathfrak{J}}^{\omega} \mid \sigma$ such that $\langle a, \sigma'' \rangle \subseteq \langle a, \sigma' \rangle$. Clearly $\langle a, \sigma'' \rangle \cap W = 0$.
(ii) → (i): Straightforward. □

Proposition 5.2. *Assume that $W \in N_{\mathfrak{J}}$. Then for every $a \in [\kappa]^{<\omega}$ and every $C \in \mathfrak{J}^+$,* II *has a winning strategy in $G_{\mathfrak{J}}(a, C, W)$.*

Proof. By Corollary 4.2. □

We will see later (Proposition 8.1) that the converse of Proposition 5.2 holds in case $\mathfrak{J}$ is almost κ-distributive.

Proposition 5.3. *$N_{\mathfrak{J}}$ is an ideal over $[\kappa]^{\omega}$.*

Proof. Immediate from Proposition 5.1. □

One way to describe $N_{\mathfrak{J}}$ is to give the values of the cardinals that are associated with it. Let us start by considering $add(N_{\mathfrak{J}})$ and $cov(N_{\mathfrak{J}})$.

Proposition 5.4. *$add(N_{\mathfrak{J}}) \geqslant \aleph_1$.*

Proof. Let $W_n \in N_{\mathfrak{J}}$ for $n < \omega$. Given $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{J}^+$, we will define a winning strategy σ for I in $G_{\mathfrak{J}}(a, C, \bigcap_{n \in \omega}([\kappa]^{\omega} - W_n))$. Let σ_0 be a winning strategy for I in $G_{\mathfrak{J}}(a, C, [\kappa]^{\omega} - W_0)$, and set $\sigma(0) = \sigma_0(0)$. Let II successively play (α_i, B_i) for $i < \omega$. Set $A_0^i = B_i$ for each $i < \omega$. For each $k \in \omega$, define σ_{k+1} and A_{k+1}^i for $k \leqslant i < \omega$ so that

(0) $A_{k+1}^i = \sigma_k((\alpha_k, A_k^k), \ldots, (\alpha_i, A_k^i))$;

(1) σ_{k+1} is a winning strategy for I in $G_{\mathfrak{J}}(a \cup \{\alpha_j: j \leqslant k\}, A_{k+1}^k, [\kappa]^{\omega} - W_{k+1})$.

Then set $\sigma((\alpha_0, B_0), \ldots, (\alpha_k, B_k)) = \sigma_{k+1}(0)$. □

Let us assume for a while that $\kappa = \omega$. If the Continuum Hypothesis holds, then of course by Proposition 5.4, $add(N_{\mathfrak{J}}) = \aleph_1 (= cov(N_{\mathfrak{J}}))$. The computation of $add(N_{\mathfrak{J}})$ is more arduous in the absence of CH, and we are only able to carry it out in some special cases (see Corollary 10.5 and Proposition 10.21). The following provides an upper bound.

Proposition 5.5. *Assume that $add(\mathfrak{J}) = \aleph_0$. Then $add(N_{\mathfrak{J}}) \leqslant \boldsymbol{b} \cap \boldsymbol{s}$.*

Proof. Pick $A_n \in \mathfrak{J}$ for $n < \omega$ with $\bigcup_{n<\omega} A_n \in \mathfrak{J}^+$. Define $k: \bigcup_{n<\omega} A_n \to \omega$ by letting $k(\alpha) = \bigcap \{n \in \omega: \alpha \in A_n\}$. Given $x \in [\kappa]^{\omega}$, we define $f_x: \omega \to \omega$ by setting $f_x(p) = 0$ in case $e_x(p) \notin \bigcup_{n<\omega} A_n$, and $f_x(p) = k(e_x(p))$ otherwise.

Let us show that $add(N_{\mathfrak{J}}) \leqslant \boldsymbol{b}$. Given $g \in \omega^{\omega}$, put

$$W_g = \left\{x \in [\kappa]^{\omega}: x \cap \bigcup_{n<\omega} A_n \in [\kappa]^{<\omega}\right\} \cup \{x \in [\kappa]^{\omega}: \{p \in \omega: f_x(p) \leqslant g(p)\} \in [\omega]^{<\omega}\}.$$

Given $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$, we define a winning strategy σ for I in $G_\mathfrak{I}(a, C, W_g)$ as follows. In case $C - \bigcup_{n<\omega} A_n \in \mathfrak{I}^+$, set $\sigma = \ulcorner C - \bigcup_{n<\omega} A_n \urcorner$. Otherwise put

$$\sigma(0) = \left(C \cap \bigcup_{n<\omega} A_n\right) - \bigcup_{m \leqslant g(|a|)} A_m \text{ and } \sigma((\alpha_0, B_0), \ldots, (\alpha_i, B_i))$$
$$= B_i - \bigcup_{m \leqslant g(|a|+i+1)} A_m.$$

Thus, $[\kappa]^\omega - W_g \in N_\mathfrak{I}$ for every $g \in \omega^\omega$. Now let $F \subseteq \omega^\omega$ be such that for every $f \in \omega^\omega$, there is a $g \in F$ with $\{n: f(n) \leqslant g(n)\} \in [\omega]^\omega$. Clearly,

$$\bigcap_{g \in F} W_g = \left\{x \in [\kappa]^\omega : x \cap \bigcup_{n<\omega} A_n \in [\kappa]^{<\omega}\right\}.$$

Let us now show that $add(N_\mathfrak{I}) \leqslant s$. For each $n < \omega$, put $E_n = A_n - \bigcup_{m<n} A_m$. Given $D \in [\omega]^\omega$, set

$$Z_D = \{x \in [\kappa]^\omega : ran(f_x) \in [\omega]^\omega \quad \text{and} \quad \{ran(f_x) \cap D, ran(f_x) - D\} \cap [\omega]^{<\omega} \neq 0\}$$

and $W_D = \{x \in [\kappa]^\omega : x \cap \bigcup_{n<\omega} A_n \in [\kappa]^{<\omega}\} \cup Z_D$. Given $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$, we define a winning strategy σ for I in $G_\mathfrak{I}(a, C, W_D)$ as follows. In case $C \cap \bigcup_{n<\omega} A_n \in \mathfrak{I}$, put $\sigma = \ulcorner C - \bigcup_{n<\omega} A_n \urcorner$. Assuming that $C \cap \bigcup_{n \in D} E_n \in \mathfrak{I}^+$, we set $\sigma(0) = C \cap \bigcup_{n \in D} E_n$ and $\sigma((\alpha_0, B_0), \ldots, (\alpha_i, B_i)) = B_i - \bigcup_{m \leqslant k(\alpha_i)} E_m$. Finally, assume that $C \cap \bigcup_{n \in D} E_n \in \mathfrak{I}$ and $C \cap \bigcup_{n \in \omega - D} E_n \in \mathfrak{I}^+$. Then set $\sigma(0) = C \cap \bigcup_{n \in \omega - D} E_n$ and $\sigma((\alpha_0, B_0), \ldots, (\alpha_i, B_i)) = B_i - \bigcup_{m \leqslant k(\alpha_i)} E_m$. Thus $[\kappa]^\omega - W_D \in N_\mathfrak{I}$ for every $D \in [\omega]^\omega$. Now let $X \subseteq [\omega]^\omega$ be such that for every $A \in [\omega]^\omega$, there is a $D \in X$ with $\{A \cap D, A - D\} \subset [\omega]^\omega$. Clearly, $\bigcap_{D \in X} W_D = \{x \in [\kappa]^\omega : x \cap \bigcup_{n<\omega} A_n \in [\kappa]^{<\omega}\}$. $\square$

Proposition 5.6. *Assume that $add(\mathfrak{I}) > \aleph_0$. Then $add(N_\mathfrak{I}) \leqslant add(\mathfrak{I})$.*

Proof. Select $A_\alpha \in \mathfrak{I}$ for $\alpha < add(\mathfrak{I})$ with $\bigcup_{\alpha < add(\mathfrak{I})} A_\alpha \in \mathfrak{I}^+$. For each $\gamma < add(\mathfrak{I})$, set $W_\gamma = \{x \in [\kappa]^\omega : x \cap \bigcup_{\alpha \leqslant \gamma} A_\alpha \in [\kappa]^\omega\}$. Given $\gamma < add(\mathfrak{I})$, $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$, $\ulcorner C - \bigcup_{\alpha \leqslant \gamma} A_\alpha \urcorner$ is a winning strategy for I in $G_\mathfrak{I}(a, C, [\kappa]^\omega - W_\gamma)$. Thus $W_\gamma \in N_\mathfrak{I}$ for all $\gamma < add(\mathfrak{I})$. However $\langle 0, \bigcup_{\alpha < add(\mathfrak{I})} A_\alpha \rangle \subseteq \bigcup_{\gamma < add(\mathfrak{I})} W_\gamma$, and so I has no winning strategy in $G_\mathfrak{I}(0, \bigcup_{\alpha < add(\mathfrak{I})} A_\alpha, [\kappa]^\omega - \bigcup_{\gamma < add(\mathfrak{I})} W_\gamma)$. Thus, $\bigcup_{\gamma < add(\mathfrak{I})} W_\gamma \notin N_\mathfrak{I}$. $\square$

It follows from the next two propositions that if $\kappa > \omega$, $\mathfrak{I}$ is κ-distributive and $add(\mathfrak{I}) = \kappa$, then $add(N_\mathfrak{I}) = cov(N_\mathfrak{I}) = \kappa$.

Proposition 5.7. *Assume that $\boldsymbol{h}_\mathfrak{I} \geqslant add(\mathfrak{I}) > \aleph_0$. Then $add(N_\mathfrak{I}) = add(\mathfrak{I})$.*

Proof. By Proposition 5.6, it is enough to show that $add(N_\mathfrak{I}) \geqslant add(\mathfrak{I})$. Thus, let ν be an infinite cardinal $< add(\mathfrak{I})$, and let $W_\gamma \in N_\mathfrak{I}$ for $\gamma < \nu$. Fix $a \in [\kappa]^{<\omega}$ and $\sigma \in \Sigma_\mathfrak{I}^\omega$. For each $\gamma < \nu$, set $S_\gamma = \{\tau \in \Sigma_\mathfrak{I}^\omega : \langle a, \tau \rangle \cap W_\gamma = 0\}$. By Proposition 5.1, each S_γ is dense in $(\Sigma_\mathfrak{I}^\omega, \leqslant)$. Hence, by Proposition 4.5, one can find $\sigma' \in \Sigma_\mathfrak{I}^\omega | \sigma$ and $\sigma_\gamma \in S_\gamma$ for $\gamma < \nu$ such that $\sigma' \in \bigcap_{\gamma<\nu} \Psi(\sigma_\gamma)$. For each $\gamma < \nu$, we clearly have $\langle a, \sigma' \rangle \subseteq \langle a, \sigma_\gamma \rangle$. Hence, $\langle a, \sigma' \rangle \cap \bigcup_{\gamma<\nu} W_\gamma = 0$. Thus, by Proposition 5.1, $\bigcup_{\gamma<\nu} W_\gamma \in N_{\mathfrak{I}}$. □

Proposition 5.8. *Assuming $\kappa > \omega$, $cov(N_{\mathfrak{I}}) \leqslant \kappa$.*

Proof. Set $W_\alpha = [\alpha]^\omega$ for every $\alpha < \kappa$. Clearly, each $W_\alpha \in N_{\mathfrak{I}}$. We have $\bigcup_{\alpha<\kappa} W_\alpha = [\kappa]^\omega$. □

Notice that by Propositions 5.4 and 5.8, we have $add(N_{\mathfrak{I}}) = cov(N_{\mathfrak{I}}) = \kappa$ in case $\kappa = \omega_1$.

Let us observe that if $cov(\mathfrak{I}) = \aleph_0$, then by the proof of Proposition 5.5, $cov(N_{\mathfrak{I}}) \leqslant \boldsymbol{b} \cap \boldsymbol{s}$. If $cov(\mathfrak{I}) > \aleph_0$ and $cov(\mathfrak{I}) = add(\mathfrak{I})$, then by the proof of Proposition 5.6, $cov(N_{\mathfrak{I}}) \leqslant add(\mathfrak{I})$.

Proposition 5.9. *Assuming $\kappa = \omega$, $cov(N_{\mathfrak{I}}) \leqslant \chi_{\mathfrak{I}}$.*

Proof. The result is immediate from the observation that for every $Q \in M_{\mathfrak{I},\omega}$, $W_Q \in N_{\mathfrak{I}}$, where $W_Q = \{E \in [\omega]^\omega \colon \{E - B \colon B \in Q\} \subseteq [\omega]^\omega\}$. □

We next turn to $non(N_{\mathfrak{I}})$. In case $\kappa > \omega$, we have by the next proposition and Proposition 1.10 that $non(N_{\mathfrak{I}}) = \kappa$ if we assume that $\kappa^{\aleph_0} = \kappa$. That assumption cannot be omitted, as is shown by Proposition 5.11.

Proposition 5.10. *$non(N_{\mathfrak{I}}) \geqslant \nu_{\mathfrak{I}}$.*

Proof. Let $W \subseteq [\kappa]^\omega$ be such that $W \notin N_{\mathfrak{I}}$. Pick $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$ so that I has no winning strategy in $G_{\mathfrak{I}}(a, C, [\kappa]^\omega - W)$. Given $Q \in M_{\mathfrak{I},\kappa}$, select $B \in Q$ with $C \cap B \in \mathfrak{I}^+$. We have that $\langle a, \ulcorner C \cap B \urcorner \rangle \cap W \neq 0$. Hence there is an $A \in W$ such that $A - B \in [\kappa]^{<\omega}$. □

Proposition 5.11. *Assume that $cov(\mathfrak{I}) = \aleph_0$. Then $\boldsymbol{d} \leqslant non(N_{\mathfrak{I}})$.*

Proof. Pick $A_n \in \mathfrak{I}$ for $n < \omega$ with $\bigcup_{n<\omega} A_n = \kappa$. Now let $W \subseteq [\kappa]^\omega$ be such that $|W| < \boldsymbol{d}$. Given $x \in W$, define $f_x \colon \omega \to \omega$ by letting $f_x(p) = \bigcap \{n \in \omega \colon e_x(p) \in A_n\}$. Find $g \in \omega^\omega$ such that $\{p \colon g(p) > f_x(p)\} \in [\omega]^\omega$ for all $x \in W$. Given $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$, we define a winning strategy σ for I in $G_{\mathfrak{I}}(a, C, [\kappa]^\omega - W)$ by letting $\sigma(0) = C - \bigcup_{n<g(|a|)} A_n$ and $\sigma((\alpha_0, B_0), \ldots, (\alpha_i, B_i)) = B_i - \bigcup_{n<g(|a|+i+1)} A_n$. □

Assuming $\kappa = \omega$, we will have from Propositions 10.10 and 5.11 that $non(N_{\mathfrak{I}}) = \boldsymbol{d}$ in case $\mathfrak{I}$ is a prime weak P-point with $cof(\mathfrak{I}) \leqslant \boldsymbol{d}$. Here is another case where we can determine the value of $non(N_{\mathfrak{I}})$.

Proposition 5.12. *Assume that $\kappa = \omega$ and $\mathfrak{I}$ is everywhere feeble. Then $non(N_{\mathfrak{I}}) = 2^{\aleph_0}$.*

Proof. By Propositions 1.13 and 5.10. □

6. $C_{\mathfrak{I}}$

The beginning of this section runs in some sense parallel to that of the previous section. This time we deal with a generalization to arbitrary ideals of the notion of completely Ramsey set, which first appeared in [29]. Here again our definition is formulated in terms of games. The basic idea, which is to work with strategies rather than with sets (of numbers), is clear from Proposition 6.1 (and from the similar Proposition 5.1). Our definition is equivalent to that of [29] in case $\mathfrak{I} = [\omega]^{<\omega}$ (see [17] and Proposition 10.23).

Let $C_{\mathfrak{I}}$ be the set of all $W \subseteq [\kappa]^{\omega}$ such that the following holds: given $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$, either I has a winning strategy in $G_{\mathfrak{I}}(a, C, W)$, or else I has a winning strategy in $G_{\mathfrak{I}}(a, C, [\kappa]^{\omega} - W)$.

Notice that $N_{\mathfrak{I}} \subseteq C_{\mathfrak{I}}$.

Proposition 6.1. *Given $W \subseteq [\kappa]^{\omega}$, the following are equivalent:*

(i) *$W \in C_{\mathfrak{I}}$.*

(ii) *For every $\langle a, \sigma\rangle \in P_{\mathfrak{I}}$, there is a $\sigma' \in \Sigma^{\omega}_{\mathfrak{I}} \,|\, \sigma$ such that either $\langle a, \sigma'\rangle \subseteq W$, or else $\langle a, \sigma'\rangle \cap W = 0$.*

Proof. An easy modification of the proof of Proposition 5.1. □

Proposition 6.2. *Let $W \in C_{\mathfrak{I}}$. Then $G_{\mathfrak{I}}(a, C, W)$ is determined for every $a \in [\kappa]^{<\omega}$ and every $C \in \mathfrak{I}^+$.*

Proof. Fix $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$. First suppose there is an $A \in \mathfrak{I}^+ \cap P(C)$ such that I has a winning strategy in $G_{\mathfrak{I}}(a, A, W)$. Then clearly I has a winning strategy in $G_{\mathfrak{I}}(a, C, W)$. Now suppose otherwise. Then for every $A \in \mathfrak{I}^+ \cap P(C)$, I has a winning strategy in $G_{\mathfrak{I}}(a, A, [\kappa]^{\omega} - W)$. Hence by Corollary 4.2, II has a winning strategy in $G_{\mathfrak{I}}(a, C, W)$. □

We will see in Section 8 that the implication of Proposition 6.2 can be reversed in case $\mathfrak{I}$ is almost κ-distributive.

We now turn to the question of the computation of $add(C_{\mathfrak{I}})$. Corollary 9.8 will show that the problem can be reduced to that of the determination of $add(N_{\mathfrak{I}})$ in case $\mathfrak{I}$ is κ-distributive. Here we present a (pretty low) lower bound and an upper bound, that are both obtained without any extra assumption on $\mathfrak{I}$.

Proposition 6.3. $add(C_3) \geqslant \aleph_0$.

Proof. Immediate from Proposition 6.1. $\square$

Proposition 6.4. *There is a* $W \subseteq [\kappa]^\omega$ *such that for every* $\langle a,\sigma\rangle \in P_3$, $\langle a,\sigma\rangle \cap W \neq 0$ *and* $\langle a,\sigma\rangle - W \neq 0$.

Proof. Pick $Q \subseteq [\kappa]^\omega$ so that

(i) $A \cap B \in [\kappa]^{<\omega}$ for all $A, B \in Q$ with $A \neq B$.

(ii) For every $C \in [\kappa]^\omega$, there is an $A \in Q$ with $A \cap C \in [\kappa]^\omega$. Given $A \in Q$, let B^A_γ for $\gamma < 2^{\aleph_0}$ be a one-to-one enumeration of $[A]^\omega$. Then select $Q^A_\gamma \subseteq [B^A_\gamma]^\omega$ for $\gamma < 2^{\aleph_0}$ so that

(0) $|Q^A_\gamma| = 2^{\aleph_0}$.

(1) $D \cap E \in [B^A_\gamma]^{<\omega}$ for all $D, E \in Q^A_\gamma$ with $D \neq E$.

(2) For every $E \in [B^A_\gamma]^\omega$, there is a $D \in Q^A_\gamma$ with $D \cap E \in [B^A_\gamma]^\omega$.

Now define $E^A_{\iota,\gamma}$ for $i < 2$ and $\gamma < 2^{\aleph_0}$ so that

(a) $E^A_{0,\gamma} \neq E^A_{1,\gamma}$.

(b) $E^A_{\iota,\gamma} \in Q^A_\gamma$.

(c) $|E^A_{j,\delta} - E^A_{\iota,\gamma}| = \aleph_0$ whenever $j < 2$ and $\delta < \gamma$.

Finally, set $W = \bigcup_{A \in Q} \{a \cup E^A_{0,\gamma} : a \in [\kappa]^{<\omega} \text{ and } \gamma < 2^{\aleph_0}\}$. Let $\langle a,\sigma\rangle \in P_3$. By Proposition 4.3, there is a $\sigma' \in \Sigma^\omega_3 | \sigma$ such that for every $D \in \langle 0,\sigma'\rangle$, $[D]^\omega \subseteq \langle 0,\sigma\rangle$. Fix $D \in \langle 0,\sigma'\rangle$ with $a \subseteq \bigcap D$, and let $A \in Q$ and $\gamma < 2^{\aleph_0}$ be such that $D \cap A = B^A_\gamma$. As $E^A_{\iota,\gamma} \subseteq D$ for each $i < 2$, we have $a \cup E^A_{0,\gamma} \in \langle a,\sigma\rangle \cap W$ and $a \cup E^A_{1,\gamma} \in \langle a,\sigma\rangle - W$. $\square$

Corollary 6.5. *There is a* $W \subseteq [\kappa]^\omega$ *such that for every* $\langle a,\sigma\rangle \in P_3$, $\langle a,\sigma\rangle \cap W \notin C_3$.

Proof. Let W be as in the statement of Proposition 6.4, and assume there is an $\langle a,\sigma\rangle \in P_3$ with $\langle a,\sigma\rangle \cap W \in C_3$. Then by Proposition 6.1, there is a $\sigma' \in \Sigma^\omega_3 | \sigma$ such that either $\langle a,\sigma'\rangle \subseteq \langle a,\sigma\rangle \cap W$, or else $\langle a,\sigma'\rangle \cap (\langle a,\sigma\rangle \cap W) = 0$. Then clearly, either $\langle a,\sigma'\rangle \subseteq W$, or else $\langle a,\sigma'\rangle \cap W = 0$, a contradiction. $\square$

Proposition 6.6. $add(N_3) \geqslant add(C_3)$.

Proof. Pick $W_\alpha \in N_3$ for $\alpha < add(N_3)$ with $\bigcup_{\alpha < add(N_3)} W_\alpha \notin N_3$. If $\bigcup_{\alpha < add(N_3)} W_\alpha \notin C_3$, we are done. So assume otherwise. Then there is an $\langle a,\sigma\rangle \in P_3$ with $\langle a,\sigma\rangle \subseteq \bigcup_{\alpha < add(N_3)} W_\alpha$. By Corollary 6.5, there exists $W \subseteq \langle a,\sigma\rangle$ with $W \notin C_3$. Now $W_\alpha \cap W \in N_3$ for every α, but $\bigcup_{\alpha < add(N_3)} (W_\alpha \cap W) \notin C_3$. $\square$

7. $(P_3, \leqslant)$

The main result of this section is Proposition 7.3, which is the key to the results in the next section. Proposition 7.3 generalizes Lemma 7.2, from which it is derived. Our proof of the lemma is based on Proposition 7.1(ii), which explains why $\mathfrak{I}$ is assumed throughout to be almost κ-distributive. It is however open whether the conclusion of Lemma 7.2 may ever hold in case $\mathfrak{I}$ is not almost κ-distributive.

Proposition 7.1. *The following are equivalent:*

(i) *$\mathfrak{I}$ is almost κ-distributive.*

(ii) *Let $\alpha \in \omega + 1$, and let $\Lambda \subseteq \Xi_\alpha$ satisfy the following: given $(f, g, h) \in \Lambda$ and $(f', g, h') \in \Xi_\alpha - \Lambda$, there is an $i < \alpha$ such that $(f(i) - f'(i)) \cup (h(i) - h'(i)) \in \mathfrak{I}^+$. Furthermore, let $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$, and assume that* II *has a winning strategy in $G_{\mathfrak{I}}(a, C, \Lambda)$. Then* I *has a winning strategy in $G_{\mathfrak{I}}(a, C', \Xi_\alpha - \Lambda)$ for every $C' \in \mathfrak{I}^+ \cap P(C)$.*

Proof. (i) $\rightarrow$ (ii): Assume that $\mathfrak{I}$ is almost κ-distributive. Let τ be a winning strategy for II in $G_{\mathfrak{I}}(a, C, \Lambda)$, and let $C' \in \mathfrak{I}^+ \cap P(C)$. We are going to define a winning strategy σ for I in $G_{\mathfrak{I}}(a, C', \Xi_\alpha - \Lambda)$. Consider a play of $G_{\mathfrak{I}}(a, C', \Xi_\alpha - \Lambda)$, with II's successive moves being (β_i, B_i) for $i < \alpha$. Using Proposition 3.1, we define for each $i < \alpha$, $A_i \in \mathfrak{I}^+$ and φ_i, ψ_i so that

(i) $\varphi_0 : \mathfrak{I}^+ \cap P(C') \rightarrow \kappa$, $\psi_0 : \mathfrak{I}^+ \cap P(C') \rightarrow \mathfrak{I}^+$ and $(\varphi_0(D), \psi_0(D)) = \tau(D)$;

(ii) $\varphi_{j+1} : \mathfrak{I}^+ \cap P(B_j \cap \psi_j(A_j)) \rightarrow \kappa$, $\psi_{j+1} : \mathfrak{I}^+ \cap P(B_j \cap \psi_j(A_j)) \rightarrow \mathfrak{I}^+$ and $(\varphi_{j+1}(D), \psi_{j+1}(D)) = \tau(A_0, \ldots, A_j, D)$;

(iii) $E_i \in dom(\varphi_i)$, and for every $\beta \in E_i$, there is an $A \in dom(\varphi_i)$ with $\varphi_i(A) = \beta$ and $E_i - \psi_i(A) \in \mathfrak{I}$;

(iv) $A_0 \subseteq C'$ and $A_{j+1} \subseteq B_j \cap \psi_j(A_j)$;

(v) $\beta_i = \varphi_i(A_i)$ and $E_i - \psi_i(A_i) \in \mathfrak{I}$.

Then set $\sigma(0) = E_0$ and $\sigma((\beta_0, B_0), \ldots, (\beta_j, B_j)) = E_{j+1}$.

(ii) $\rightarrow$ (i): Assume that (ii) holds, and let $Q_\beta \in M_{\mathfrak{I},\kappa}$ for $\beta < \kappa$. Define $\Lambda \subseteq \Xi_1$ by letting $(f, g, h) \in \Lambda$ if and only if for every $D \in Q_{g(0)}$, $h(0) - D \in \mathfrak{I}^+$. Given $C \in \mathfrak{I}^+$, II has an obvious winning strategy in $G_{\mathfrak{I}}(0, C, \Lambda)$. Hence, I has a winning strategy in $G_{\mathfrak{I}}(0, C, \Xi_1 - \Lambda)$. Then for every $\beta \in \sigma(0)$, there is a $D \in Q_\beta$ such that $\sigma(0) - D \in \mathfrak{I}$. Therefore, $\mathfrak{I}$ is almost κ-distributive. $\square$

Given $\sigma \in \Sigma_{\mathfrak{I}}^\omega$, $\alpha \in \omega$ and $(g, h) \in P_{a,\sigma}^\alpha$, we define $\sigma | (g, h) \in \Sigma_{\mathfrak{I}}^\omega$ as follows. In case $\alpha = 0$, put $\sigma | (g, h) = \sigma$. Otherwise let

$$\sigma | (g, h)(0) = \sigma((g(0), h(0)), \ldots, (g(\alpha - 1), h(\alpha - 1)));$$
$$\sigma | (g, h)(\beta_0, B_0) = \sigma((g(0), h(0)), \ldots, (g(\alpha - 1), h(\alpha - 1)), (\beta_0, B_0));$$

etc.

We will often write $\sigma | (g(0), h(0))$ instead of $\sigma | (g, h)$ in case $\alpha = 1$.

Given $\langle a, \sigma \rangle, \langle a', \sigma' \rangle \in P_{\mathfrak{I}}$, we put $\langle a', \sigma' \rangle \leqslant \langle a, \sigma \rangle$ if and only if there is a $(g, h) \in \bigcup_{\alpha \in \omega} P_{a,\sigma}^\alpha$ such that $a' = a \cup ran(g)$ and $\sigma'' \in \Phi(\sigma | (g, h))$, where $\sigma'' \in \Sigma_{\mathfrak{I}}^\omega$ is defined by letting $\sigma''(0) = \{\beta \in \sigma'(0)\colon a \subseteq \beta\}$ and

$$\sigma''((\beta_0, B_0), \ldots, (\beta_i, B_i)) = \sigma'((\beta_0, B_0), \ldots, (\beta_i, B_i)).$$

It is not difficult to check that $\leqslant$ is a transitive relation over $P_{\mathfrak{J}}$. We observe that $\langle a', \sigma' \rangle \leqslant \langle a, \sigma \rangle$ implies $\langle a', \sigma' \rangle \sqsubseteq \langle a, \sigma \rangle$. Notice that $\langle a, \sigma' \rangle \leqslant \langle a, \sigma \rangle$ whenever $\sigma' \leqslant \sigma$.

Lemma 7.2. *Assume that $\mathfrak{J}$ is almost κ-distributive. Let $X \subseteq P_{\mathfrak{J}}$ be open and dense in $(P_{\mathfrak{J}}, \leqslant)$, and let $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{J}^+$. Then there is a $\sigma \in \Sigma_{\mathfrak{J}}^{\omega} | \ulcorner C \urcorner$ with the following property: for every $(g, h) \in P_{a,\sigma}^{\omega}$, there is an $i < \omega$ such that*

$$\langle a \cup ran(g|i), \sigma|(g|i, h|i) \rangle \in X.$$

Proof. Define $\Lambda \subseteq \Xi_\omega$ by letting $(f, g, h) \in \Lambda$ if and only if for some $i < \omega$, there is a $\sigma \in \Sigma_{\mathfrak{J}}^{\omega} | \ulcorner f(i) \urcorner$ such that $\langle a \cup ran(g|i), \sigma \rangle \in X$. We claim that II has no winning strategy in $G_{\mathfrak{J}}(a, C, \Lambda)$. Suppose otherwise. Let $(f, g, h) \in \Lambda$ and $(f', g, h') \in \Xi_\omega$ be given such that for every $i < \omega, f(i) - f'(i) \in \mathfrak{J}$. Select $i < \omega$ and $\sigma_0 \in \Sigma_{\mathfrak{J}}^{\omega} | \ulcorner f(i) \urcorner$ such that $\langle a \cup ran(g|i), \sigma_0 \rangle \in X$. Define $\sigma_1 \in \Sigma_{\mathfrak{J}}^{\omega}$ by letting $\sigma_1(0) = \sigma_0(0) \cap f'(i)$ and $\sigma_1((\alpha_0, B_0), \ldots, (\alpha_i, B_i)) = \sigma_0((\alpha_0, B_0), \ldots, (\alpha_i, B_i))$. Clearly, $\sigma_1(0) \subseteq f'(i)$ and $\langle a \cup ran(g|i), \sigma_1 \rangle \in X$. Thus, $(f', g, h') \in \Lambda$. Hence by Proposition 7.1, there is a winning strategy σ_2 for I in $G_{\mathfrak{J}}(a, C, \Xi_\omega - \Lambda)$. Since, X is dense in $(P_{\mathfrak{J}}, \leqslant)$, there are $(t, v) \in \bigcup_{i \in \omega} P_{a,\sigma}^i$ and $\sigma_3 \in \Phi(\sigma_2 | (t, v))$ such that $\langle a \cup ran(t), \sigma_3 \rangle \in X$, which yields a contradiction. By Lemma 4.6, $G_{\mathfrak{J}}(a, C, \Lambda)$ is determined. Hence there is a winning strategy σ_4 for I in $G_{\mathfrak{J}}(a, C, \Lambda)$. Let us now define $\sigma \in \Sigma_{\mathfrak{J}}^{\omega}$. Let II successively play (α_i, B_i) for $i < \omega$. Proceed as follows. First assume that there is a $\tau \in \Sigma_{\mathfrak{J}}^{\omega} | \ulcorner \sigma_4(0) \urcorner$ with $\langle a, \tau \rangle \in X$. Then put $\sigma(0) = \tau(0)$ and

$$\sigma((\alpha_0, B_0), \ldots, (\alpha_i, B_i)) = \tau((\alpha_0, B_0), \ldots, (\alpha_i, B_i)).$$

Now assume otherwise. We set $\sigma(0) = \sigma_4(0)$. In case there is a $\tau \in \Sigma_{\mathfrak{J}}^{\omega} | \ulcorner \sigma_4(\alpha_0, B_0) \urcorner$ with $\langle a \cup \{\alpha_0\}, \tau \rangle \in X$, set $\sigma(\alpha_0, B_0) = \tau(0)$ and

$$\sigma((\alpha_0, B_0), (\alpha_1, B_1), \ldots, (\alpha_i, B_i)) = \tau((\alpha_1, B_1), \ldots, (\alpha_i, B_i)).$$

Otherwise put $\sigma(\alpha_0, B_0) = \sigma_4(\alpha_0, B_0)$; etc. σ is easily seen to be as desired. $\square$

Let us observe the following. Let $\sigma \in \Sigma_{\mathfrak{J}}^{\omega}$, and let $\sigma' \in \Phi(\sigma)$. Furthermore, let $\rho\colon \bigcup_{i < \omega} P_{0,\sigma'}^{i+1} \to \mathfrak{J}^+$ be such that for every $i < \omega$ and every $(g, h) \in P_{0,\sigma'}^{i+1}$,

$$\sigma'((g(0), h(0)), \ldots, (g(i), h(i))) \subseteq \sigma((g(0), \rho(g|1, h|1)), \ldots, (g(i), \rho(g, h))).$$

Then for every $i < \omega$ and every $(g, h) \in P_{0,\sigma'}^{i+1}$, $\sigma'|(g, h) \in \Phi(\sigma|(g, t))$, where for every $j \leqslant i$, $t(j) = \rho(g|j+1, h|j+1)$.

Proposition 7.3. *Assume that $\mathfrak{J}$ is almost κ-distributive. Let $X \subseteq P_{\mathfrak{J}}$ be open and dense in $(P_{\mathfrak{J}}, \leqslant)$, and let $a \in [\kappa]^{<\omega}$ and $\sigma \in \Sigma_{\mathfrak{J}}^{\omega}$. Then there is a $\sigma' \in \Sigma_{\mathfrak{J}}^{\omega} | \sigma$ with the following* *property: for every* $(g,h) \in P^{\omega}_{a,\sigma'}$, *there is an* $i < \omega$ *such that* $\langle a \cup ran(g \mid i), \sigma' \mid (g \mid i, h \mid i)\rangle \in X$.

Proof. By Lemma 7.2, one can find $\tau \in \Sigma^{\omega}_{\mathfrak{I}} \mid \ulcorner \sigma(0) \urcorner$ such that for every $(g,h) \in P^{\omega}_{a,\tau}$, there is an $i < \omega$ with $\langle a \cup ran(g \mid i), \tau \mid (g \mid i, h \mid i)\rangle \in X$. We define $\sigma' \in \Sigma^{\omega}_{\mathfrak{I}}$ as follows. Put $\sigma'(0) = \tau(0)$. Suppose II's successive moves are (α_i, B_i) for $i < \omega$. Let A_i and E_i for $i < \omega$ be such that

(0) $A_0 = \tau(0)$ and $E_0 = \sigma(0)$;

(1) $E_{j+1} = \sigma((\alpha_0, B_0), \ldots, (\alpha_j, B_j))$;

(2) $A_{j+1} = \tau((\alpha_0, E_1), \ldots, (\alpha_j, E_{j+1}))$.

Set $\sigma'((\alpha_0, B_0), \ldots, (\alpha_j, B_j)) = A_{j+1}$. Define $h, h' \in (\mathfrak{I}^+)^{\omega}$ and $g \in \kappa^{\omega}$ by letting $h(i) = E_{i+1}$, $h'(i) = B_i$ and $g(i) = \alpha_i$. For each $i < \omega$, $\langle a \cup ran(g \mid i), \tau \mid (g \mid i, h \mid i)\rangle \in X$ implies $\langle a \cup ran(g \mid i), \sigma' \mid (g \mid i, h \mid i)\rangle \in X$. Thus σ' is as desired. □

We conclude the section by showing that in case $add(\mathfrak{I}) = \kappa$, two quantifiers can be exchanged in the conclusion of Proposition 7.3, which yields a stronger result. The proof is based on the following modified version of Lemma 7.2.

Lemma 7.4. *Assume that* $\mathfrak{I}$ *is* κ*-distributive and* $add(\mathfrak{I}) = \kappa$. *Let* $X \subseteq P_{\mathfrak{I}}$ *be open and dense in* $(P_{\mathfrak{I}}, \leqslant)$, *and let* $\sigma \in \Sigma^{\omega}_{\mathfrak{I}}$ *and* $\beta \in \kappa$. *Then there is a* $\sigma' \in \Sigma^{\omega}_{\mathfrak{I}} \mid \sigma$ *with the following property: for every* $a \in [\beta]^{<\omega}$ *and every* $(g,h) \in P^{\omega}_{a,\sigma'}$, *there is an* $i < \omega$ *such that* $\langle a \cup ran(g \mid i), \sigma' \mid (g \mid i, h \mid i)\rangle \in X$.

Proof. For each $a \in [\beta]^{<\omega}$, let S_a be the set of all $\tau \in \Sigma^{\omega}_{\mathfrak{I}}$ such that for every $(g,h) \in P^{\omega}_{a,\tau}$, there is an $i < \omega$ with $\langle a \cup ran(g \mid i), \tau \mid (g \mid i, h \mid i)\rangle \in X$. By Proposition 7.3, each S_a is dense in $(\Sigma^{\omega}_{\mathfrak{I}}, \leqslant)$. Now use Proposition 4.5 to find $\sigma' \in \Sigma^{\omega}_{\mathfrak{I}} \mid \sigma \cap (\bigcap_{a \in [\beta]^{<\omega}} \bigcup_{\tau \in S_a} \Psi(\tau))$. It is not difficult to verify that σ' is as desired. □

Proposition 7.5. *Assume that* $\mathfrak{I}$ *is* κ*-distributive and* $add(\mathfrak{I}) = \kappa$. *Let* $X \subseteq P_{\mathfrak{I}}$ *be open and dense in* $(P_{\mathfrak{I}}, \leqslant)$, *and let* $\sigma \in \Sigma^{\omega}_{\mathfrak{I}}$. *Then there is a* $\sigma' \in \Sigma^{\omega}_{\mathfrak{I}} \mid \sigma$ *such that the following holds: for every* $a \in [\kappa]^{<\omega}$ *and every* $(g,h) \in P^{\omega}_{a,\sigma'}$, *there is an* $i < \omega$ *with* $\langle a \cup ran(g \mid i), \sigma' \mid (g \mid i, h \mid i)\rangle \in X$.

Proof. We define σ' as follows. We put $\sigma'(0) = \sigma(0)$. Let II play (α_0, B_0). Use Lemma 7.4 to find $\tau \in \Sigma^{\omega}_{\mathfrak{I}} \mid (\sigma \mid (\alpha_0, B_0))$ such that for every $b \in [\alpha_0 + 1]^{<\omega}$ and every $(g,h) \in P^{\omega}_{b,\tau}$, there is an $i < \omega$ with $\langle b \cup ran(g \mid i), \tau \mid (g \mid i, h \mid i)\rangle \in X$. We let $\sigma' \mid (\alpha_0, B_0) = \tau$. □

8. $N_{\mathfrak{I}}$ and $C_{\mathfrak{I}}$

We will now build upon our preliminary work to obtain our main combinatorial results: characterizations of $N_{\mathfrak{I}}$ and $C_{\mathfrak{I}}$, and closure of $C_{\mathfrak{I}}$ under operation A (see

Section 0 for a definition of this property). The results of this section will all be refined in the next section, but there we will usually assume more.

It stands out that (iii) in Proposition 8.3 (or in Proposition 8.1) lacks a certain degree of naturalness, as the definition of $\leqslant$ is too cumbersome. It would clearly be desirable to replace $\leqslant$ by a transitive relation with a simpler definition. Our solution (in the next section) to this problem will consist in modifying both the definition of $P_{\mathfrak{I}}$ and that of $\leqslant$. The fact remains that $\leqslant$ does the job. We feel moreover that the equivalence between (i) and (iii) in Proposition 8.3 (or in Proposition 8.1) is in some sense more basic than the more elegant results of Sections 9 and 10.

Another (major) problem is that we are unable to determine whether $P_{\mathfrak{I}} \subseteq C_{\mathfrak{I}}$ holds in all cases.

As for the assumption (of almost κ-distributivity) that is used throughout this section, we do not know whether it can be weakened.

Proposition 8.1. *Assume that $\mathfrak{I}$ is almost κ-distributive. Then given $W \subseteq [\kappa]^{\omega}$, the following are equivalent:*

(i) $W \in N_{\mathfrak{I}}$.

(ii) *For every $a \in [\kappa]^{<\omega}$ and every $C \in \mathfrak{I}^+$, II has a winning strategy in $G_{\mathfrak{I}}(a, C, W)$.*

(iii) $\{\langle a,\sigma\rangle \in P_{\mathfrak{I}}\colon \langle a,\sigma\rangle \cap W = 0\}$ *is dense in* $(P_{\mathfrak{I}}, \leqslant)$.

Proof. (i) $\to$ (ii): By Proposition 5.2.

(ii) $\to$ (iii): Assume (ii), and fix $\langle a,\sigma\rangle \in P_{\mathfrak{I}}$. By Proposition 7.1, there is a $\sigma' \in \Sigma_{\mathfrak{I}}^{\omega} | \ulcorner \sigma(0) \urcorner$ with $\langle a,\sigma'\rangle \cap W = 0$. By Proposition 4.4, there is a $\sigma'' \in \Sigma_{\mathfrak{I}}^{\omega} | \sigma$ with $\langle a,\sigma''\rangle \subseteq \langle a,\sigma'\rangle$. Clearly, $\langle a,\sigma''\rangle \leqslant \langle a,\sigma\rangle$, and $\langle a,\sigma''\rangle \cap W = 0$.

(iii) $\to$ (i): Set $X = \{\langle a,\sigma\rangle \in P_{\mathfrak{I}}\colon \langle a,\sigma\rangle \cap W = 0\}$. X is clearly open in $(P_{\mathfrak{I}}, \leqslant)$. Let us assume that X is dense in $(P_{\mathfrak{I}}, \leqslant)$. Fix $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$. By Lemma 7.2, one can find $\sigma \in \Sigma_{\mathfrak{I}}^{\omega} | \ulcorner C \urcorner$ such that for every $(g,h) \in P_{a,\sigma}^{\omega}$, there is an $i < \omega$ with $\langle a \cup ran(g | i), \sigma | (g | i, h | i)\rangle \in X$. Then σ is a winning strategy for I in $G_{\mathfrak{I}}(a, C, [\kappa]^{\omega} - W)$. Thus, $W \in N_{\mathfrak{I}}$. $\square$

We again remark that an exchange of quantifiers (this time in (ii) of Proposition 5.1) is possible in case $add(\mathfrak{I}) = \kappa$.

Proposition 8.2. *Assume that $\mathfrak{I}$ is κ-distributive and $add(\mathfrak{I}) = \kappa$. Then given $W \subseteq [\kappa]^{\omega}$, $W \in N_{\mathfrak{I}}$ if and only if for every $\sigma \in \Sigma_{\mathfrak{I}}^{\omega}$, there is a $\sigma' \in \Sigma_{\mathfrak{I}}^{\omega} | \sigma$ with $\bigcup_{a \in [\kappa]^{<\omega}} \langle a,\sigma'\rangle \cap W = 0$.*

Proof. By Propositions 5.1, 8.1 and 7.5. $\square$

Proposition 8.3. *Assume that $\mathfrak{I}$ is almost κ-distributive. Then given $W \subseteq [\kappa]^{\omega}$, the following are equivalent:*

(i) $W \in C_{\mathfrak{I}}$.

(ii) *For every* $a \in [\kappa]^{<\omega}$ *and every* $C \in \mathfrak{I}^+$, $G_{\mathfrak{I}}(a, C, W)$ *is determined.*
(iii) $\{\langle a, \sigma\rangle \in P_{\mathfrak{I}}$: $\langle a, \sigma\rangle \subseteq W$ *or* $\langle a, \sigma\rangle \cap W = 0\}$ *is dense in* $(P_{\mathfrak{I}}, \leqslant)$.

Proof. (i) → (ii): By Proposition 6.2.

(ii) → (iii): The proof is essentially the same as that of (ii) → (iii) in Proposition 8.1.

(iii) → (i): Assume (iii), and fix $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$. By Lemma 7.2, one can find $\sigma \in \Sigma_{\mathfrak{I}}^{\omega} | \ulcorner C \urcorner$ such that for every $(g, h) \in P_{a,\sigma}^{\omega}$, there is an $i < \omega$ with $\langle a \cup ran(g | i), \sigma | (g | i, h | i)\rangle \in X$. We claim that $G_{\mathfrak{I}}(a, \sigma(0), W)$ is determined. Let us assume that I has no winning strategy in $G_{\mathfrak{I}}(a, \sigma(0), W)$. We will define a winning strategy τ for II in $G_{\mathfrak{I}}(a, \sigma(0), W)$. Consider a play of $G_{\mathfrak{I}}(a, \sigma(0), W)$ where I's successive moves are A_i for $i < \omega$. Define $g \in \{\alpha \in \kappa\colon a \subseteq \alpha\}^{\omega}$, $h \in (\mathfrak{I}^+)^{\omega}$ and $h' \in (\mathfrak{I}^+)^{\omega}$ so that for every $i < \omega$,

(0) $g(i) \in A_i$;
(1) $h'(i) \subseteq A_i - (g(i) + 1)$;
(2) I has no winning strategy in $G_{\mathfrak{I}}(a \cup \{g(j)\colon j \leqslant i\}, h'(i), W)$;
(3) $h(i) = \sigma((g(0), h'(0)), \ldots, (g(i), h'(i)))$.

Then set $\tau(A_0, \ldots, A_i) = (g(i), h(i))$. Suppose $a \cup ran(g) \in W$. Since $(g, h') \in P_{a,\sigma}^{\omega}$, one can find $i < \omega$ such that $\langle a \cup ran(g | i), \sigma | (g | i, h' | i)\rangle \in X$. Clearly $\langle a \cup ran(g | i), \sigma | (g | i, h' | i)\rangle \subseteq W$. We have $i > 0$, since otherwise σ would be a winning strategy for I in $G_{\mathfrak{I}}(a, \sigma(0), W)$. But then $\sigma | (g | i, h' | i)$ is a winning strategy for I in $G_{\mathfrak{I}}(a \cup ran(g | i), h'(i-1), W)$, a contradiction.

By the claim and Proposition 7.1, I has a winning strategy in one of the following two games: $G_{\mathfrak{I}}(a, \sigma(0), W)$, $G_{\mathfrak{I}}(a, \sigma(0), [\kappa]^{\omega} - W)$. Then clearly, either I has a winning strategy in $G_{\mathfrak{I}}(a, C, W)$, or else I has a winning strategy in $G_{\mathfrak{I}}(a, C, [\kappa]^{\omega} - W)$. Hence $W \in C_{\mathfrak{I}}$. □

The following should be compared with Proposition 6.1.

Proposition 8.4. *Assume that* $\mathfrak{I}$ *is* κ*-distributive and* $add(\mathfrak{I}) = \kappa$. *Then given* $W \subseteq [\kappa]^{\omega}$, *the following are equivalent*:

(i) $W \in C_{\mathfrak{I}}$.
(ii) *For every* $\sigma \in \Sigma_{\mathfrak{I}}^{\omega}$, *there is a* $\sigma' \in \Sigma_{\mathfrak{I}}^{\omega} | \sigma$ *with the following property*: *given* $(g, h) \in P_{0,\sigma'}^{1}$ *and* $a \in [g(0) + 1]^{<\omega}$, *either* $\langle a, \sigma' | (g, h)\rangle \subseteq W$, *or else* $\langle a, \sigma' | (g, h)\rangle \cap W = 0$.

Proof. (i) → (ii): Assume that $W \in C_{\mathfrak{I}}$. Given $\sigma \in \Sigma_{\mathfrak{I}}^{\omega}$, we define σ' as follows. We let $\sigma'(0) = \sigma(0)$. Now let II play (α_0, B_0). For each $a \in [\alpha_0 + 1]^{<\omega}$, let S_a be the set of all $\tau \in \Sigma_{\mathfrak{I}}^{\omega}$ such that either $\langle a, \tau\rangle \subseteq W$, or else $\langle a, \tau\rangle \cap W = 0$. Use Proposition 4.5 to find $\sigma'' \in \Sigma_{\mathfrak{I}}^{\omega} | (\sigma | (\alpha_0, B_0)) \cap (\bigcap_{a \in [\alpha_0 + 1]^{<\omega}} \bigcup_{\tau \in S_a} \Psi(\tau))$. Then put $\sigma' | (\alpha_0, B_0) = \sigma''$.

(ii) → (i): Straightforward. □

Let us observe that Proposition 9.7 will supersede the next proposition in the case $\kappa = \omega$.

Proposition 8.5. *Assume $\mathfrak{I}$ is almost κ-distributive. Let μ be a cardinal with $0 < \mu < \aleph_1 \cdot add(\mathfrak{I})$, and let $W_x \in C_{\mathfrak{I}}$ for $x \in \bigcup_{n \in \omega} \mu^n$. Then $\bigcup_{f \in \mu^\omega} \bigcap_{n \in \omega} W_{f|n} \in C_{\mathfrak{I}}$.*

Proof. Let $n \in \omega$, and let $x \in \mu^n$. Put $T_x = \bigcup \{\bigcap_{k \in \omega} W_{f|k}: f \in \mu^\omega \text{ and } x \subset f\}$. Clearly, $T_x = \bigcup_{\gamma \in \mu} T_{x \cup \{(n,\gamma)\}}$. Notice that $T_0 = \bigcup_{f \in \mu^\omega} \bigcap_{n \in \omega} W_{f|n}$. We define $Y_x \subseteq P_{\mathfrak{I}}$ by letting $\langle a, \sigma \rangle \in Y_x$ if and only if $\langle a', \sigma' \rangle \cap T_x \neq 0$ for all $\langle a', \sigma' \rangle \in P_{\mathfrak{I}}$ with $\langle a', \sigma' \rangle \leqslant \langle a, \sigma \rangle$. We then let X_x be the set of all $\langle a, \sigma \rangle \in P_{\mathfrak{I}}$ such that either $\langle a, \sigma \rangle \cap T_x = 0$, or else $\langle a, \sigma \rangle \in P(W_x) \cap \bigcup_{\gamma \in \mu} Y_{x \cup \{(n,\gamma)\}}$.

Let us show that X_x is dense in $(P_{\mathfrak{I}}, \leqslant)$. Thus let $\langle a, \sigma \rangle \in P_{\mathfrak{I}}$. Let us first suppose that there are $\gamma \in \mu$ and $\langle a', \sigma' \rangle \leqslant \langle a, \sigma \rangle$ with $\langle a', \sigma' \rangle \in Y_{x \cup \{(n,\gamma)\}}$. By Proposition 6.1, there is a $\sigma'' \in \Sigma_{\mathfrak{I}}^\omega | \sigma'$ such that either $\langle a', \sigma'' \rangle \subseteq W_x$, or else $\langle a', \sigma'' \rangle \cap W_x = 0$. We have $\langle a', \sigma'' \rangle \cap T_{x \cup \{(n,\gamma)\}} \neq 0$ and $T_{x \cup \{(n,\gamma)\}} \subseteq W_x$. Thus, $\langle a', \sigma'' \rangle \in P(W_x) \cap Y_{x \cup \{(n,\gamma)\}}$. Moreover, $\langle a', \sigma'' \rangle \leqslant \langle a, \sigma \rangle$. Let us now suppose that for every $\delta \in \mu$ and every $\langle b, \eta \rangle \leqslant \langle a, \sigma \rangle$, $\langle b, \eta \rangle \notin Y_{x \cup \{(n,\delta)\}}$. Given $\delta \in \mu$, define $Z_x^\delta \subseteq P_{\mathfrak{I}}$ by letting $\langle b, \eta \rangle \in Z_x^\delta$ if and only if either $\langle b, \eta \rangle$ and $\langle a, \sigma \rangle$ are incompatible in $(P_{\mathfrak{I}}, \leqslant)$, or else $\langle b, \eta \rangle \leqslant \langle a, \sigma \rangle$ and $\langle b, \eta \rangle \cap T_{x \cup \{(n,\delta)\}} = 0$. Z_x^δ is easily seen to be dense and open in $(P_{\mathfrak{I}}, \leqslant)$. Hence given $\langle b, \eta \rangle \leqslant \langle a, \sigma \rangle$, there is by Proposition 7.3, $\eta' \in \Sigma_{\mathfrak{I}}^\omega | \eta$ with the following property: for every $(g, h) \in P_{b,\eta'}^\omega$, there is an $i < \omega$ such that $\langle b \cup ran(g|i), \eta' | (g|i, h|i) \rangle \in Z_x^\delta$. It is clear that $\langle b, \eta' \rangle \cap T_{x \cup \{(n,\delta)\}} = 0$. Thus, setting $S_\delta = \{\tau \in \Sigma_{\mathfrak{I}}^\omega: \langle a, \tau \rangle \cap T_{x \cup \{(n,\delta)\}} = 0\}$, we have that $S_\delta \cap \Sigma_{\mathfrak{I}}^\omega | \eta \neq 0$ for every $\eta \in \Sigma_{\mathfrak{I}}^\omega | \sigma$. Hence, in case $add(\mathfrak{I}) > \aleph_0$ Propositions 4.5 and 3.3 can be used to find $\rho \in \Sigma_{\mathfrak{I}}^\omega | \sigma \cap (\bigcap_{\delta < \mu} \bigcup_{\tau \in S_\delta} \Psi(\tau))$. Clearly, $\langle a, \rho \rangle \cap T_x = 0$. In case $add(\mathfrak{I}) = \aleph_0$ we define $\tau \in \Sigma_{\mathfrak{I}}^\omega | \sigma$ as follows. Let $\pi_0 \in \Sigma_{\mathfrak{I}}^\omega | \sigma$ be such that $\langle a, \pi_0 \rangle \cap T_{x \cup \{(n,0)\}} = 0$, and set $\tau(0) = \pi_0(0)$. Let II successively play (β_i, C_i) for $i < \omega$. Define π_{k+1} for $k < \omega$ so that

(a) in case $k + 1 \geqslant \mu$, $\pi_{k+1} = \pi_k | (\beta_k, C_k)$;

(b) in case $k + 1 < \mu$, $\pi_{k+1} \leqslant \pi_k | (\beta_k, C_k)$ and

$$\langle a \cup \{\beta_i: i \leqslant k\}, \pi_{k+1} \rangle \cap T_{x \cup \{(n,k+1)\}} = 0.$$

Then set $\tau((\beta_0, C_0), \ldots, (\beta_k, C_k)) = \pi_{k+1}(0)$. Clearly, $\langle a, \tau \rangle \cap T_x = 0$.

Now fix $\langle d, \theta \rangle \in P_{\mathfrak{I}}$, and assume that $\langle d', \theta' \rangle \cap T_0 \neq 0$ whenever $\langle d', \theta' \rangle \leqslant \langle d, \theta \rangle$. We are going to define $\chi \in \Sigma_{\mathfrak{I}}^\omega | \theta$. Let (α_i, B_i) for $i < \omega$ be the successive moves of II. Using Proposition 7.3, we define σ_j, τ_j, u_j and v_j for $j < \omega$, $p \in \mu^\omega$ and $q \in \omega^\omega$ so that

(0) $\sigma_0 \in \Sigma_{\mathfrak{I}}^\omega | \theta \cap \Sigma_{\mathfrak{I}}^\omega | \ulcorner \{\beta \in \kappa: d \subseteq \beta\} \urcorner$;

(1) $\sigma_{j+1} \in \Sigma_{\mathfrak{I}}^\omega | \tau_j$;

(2) $q(0) = 0$ and $q(j) < q(j+1)$;

(3) $u_j: q(j+1) - q(j) \to \kappa$ is such that $u_j(r) = \alpha_{q(j)+r}$;

(4) $v_j: q(j+1) - q(j) \to \mathfrak{I}^+$ is such that $v_j(r) = B_{q(j)+r}$;

(5) $\tau_j = \sigma_j | (u_j, v_j)$;

(6) for every $(g, h) \in P_{d \cup \{\alpha_t: t < q(j)\}, \sigma_j}^\omega$, there is an $i < \omega$ with

$$\langle d \cup \{\alpha_t: t < q(j)\} \cup ran(g|i), \sigma_j | (g|i, h|i) \rangle \in X_{p|j};$$

(7) $\langle d \cup \{\alpha_t: t < q(j+1)\}, \tau_j \rangle \in P(W_{p|j}) \cap Y_{p|(j+1)}$;

(8) if $q(j) < i < q(j+1)$, then

$$\langle d \cup \{\alpha_t: t < i\}, \sigma_j | (u_j | (i - q(j)), v_j | (i - q(j)))\rangle \notin P(W_{p|j}) \cap \bigcup_{\xi \in \mu} Y_{(p|j) \cup \{(j,\xi)\}}.$$

We put $\chi(0) = \sigma_0(0)$, and $\chi((\alpha_0, B_0), \ldots, (\alpha_i, B_i)) = \sigma_j((\alpha_{q(j)}, B_{q(j)}), \ldots, (\alpha_i, B_i))$ whenever $q(j) \leqslant i < q(j+1)$. It is clear that $d \cup \{\alpha_i: i < \omega\} \in \bigcap_{j<\omega} W_{p|j}$. Thus $\langle d, \chi\rangle \subseteq T_0$. Hence by Proposition 8.3, $T_0 \in C_{\mathfrak{I}}$. □

Corollary 8.6. *Assume that $\mathfrak{I}$ is almost κ-distributive and $add(\mathfrak{I}) > \aleph_0$. Then $add(C_{\mathfrak{I}}) = add(N_{\mathfrak{I}})$.*

Proof. By Propositions 6.6, 3.3, 5.7 and 8.5. □

9. $G^*_{\mathfrak{I}}(a, C, W)$

We will now see that the results of Section 8 can be recast in more elegant formulations. We will however have to leave out the case where our ideal $\mathfrak{I}$ is almost κ-distributive but not κ-distributive, and such that $add(\mathfrak{I}) = \aleph_0$. We recall that we do not know whether such ideals can ever exist.

We first introduce the following new game.

Let α be an ordinal with $\omega \geqslant \alpha > 0$. Given $a \in [\kappa]^{<\omega}$, $C \in \mathfrak{I}^+$ and $\Theta \subseteq (\mathfrak{I}^+)^\alpha \times \kappa^\alpha$, we define the game $G^*_{\mathfrak{I}}(a, C, \Theta)$ as follows. Each player makes α moves. I (respectively II) thus constructs $f \in (\mathfrak{I}^+)^\alpha$ (resp. $g \in \kappa^\alpha$) so that

(0) $f(0) \in \mathfrak{I}^+ \cap P(C)$;

(1) $f(\delta) \subseteq f(\eta)$ for $\eta < \delta$;

(2) $f(0) - f(\delta) \in \mathfrak{I}$;

(3) $a \subseteq g(0)$;

(4) $g(\delta) \in f(\delta)$;

(5) $g(\delta) < \bigcap f(\delta + 1)$.

I wins if $(f, g) \in \Theta$.

Given $W \subseteq [\kappa]^{|a|+\alpha}$, we let $G^*_{\mathfrak{I}}(a, C, W)$ stand for $G^*_{\mathfrak{I}}(a, C, \{(f,g) \in (\mathfrak{I}^+)^\alpha \times \kappa^\alpha: a \cup ran(g) \in W\})$.

We let $\Sigma^{*\omega}_{\mathfrak{I}}$ be the set of all strategies for I in $G^*_{\mathfrak{I}}(0, \kappa, (\mathfrak{I}^+)^\omega \times \kappa^\omega)$.

Let $a \in [\kappa]^{<\omega}$, and let $\sigma \in \Sigma^{*\omega}_{\mathfrak{I}}$. We let $\langle a, \sigma\rangle^*$ be the set of all $x \in [\kappa]^\omega$ such that $a = \{e_x(i): i < |a|\}$, $e_x(|a|) \in \sigma(0)$ and for every $j < \omega$, $e_x(|a| + j + 1) \in \sigma(e_x(|a|), \ldots, e_x(|a| + j))$.

We set $P^*_{\mathfrak{I}} = \{\langle a, \sigma\rangle^*: a \in [\kappa]^{<\omega}$ *and* $\sigma \in \Sigma^{*\omega}_{\mathfrak{I}}\}$.

Given $\sigma, \sigma' \in \Sigma^{*\omega}_{\mathfrak{I}}$, we let $\sigma' \leqslant \sigma$ just in case $\sigma'(0) \subseteq \sigma(0)$ and $\sigma'(\alpha_0, \ldots, \alpha_i) \subseteq \sigma(\alpha_0, \ldots, \alpha_i)$.

Observe that $\sigma' \leqslant \sigma$ if and only if $\langle 0, \sigma'\rangle^* \subseteq \langle 0, \sigma\rangle^*$.

We define $\varepsilon_0 : \Sigma_{\mathfrak{I}}^{*\omega} \to \Sigma_{\mathfrak{I}}^{\omega}$ by letting $(\varepsilon_0(\sigma))(0) = \sigma(0)$ and

$$(\varepsilon_0(\sigma))((\alpha_0, B_0), \ldots, (\alpha_i, B_i)) = \sigma(\alpha_0, \ldots, \alpha_i) \cap B_i.$$

Notice that $\langle a, \sigma \rangle^* = \langle a, \varepsilon_0(\sigma) \rangle$ for every $a \in [\kappa]^{<\omega}$.

We just saw that it is easy to associate an element of $\Sigma_{\mathfrak{I}}^{\omega}$ with any given $\sigma \in \Sigma_{\mathfrak{I}}^{*\omega}$. Our present goal is, conversely, to define a function from $\Sigma_{\mathfrak{I}}^{\omega}$ to $\Sigma_{\mathfrak{I}}^{*\omega}$. That will require some preparatory work. We start with the following lemma.

Lemma 9.1. *Assume that $\mathfrak{I}$ is almost κ-distributive. Let $A \in \mathfrak{I}^+$, $n \in \omega - 1$ and $Q_b \in M_{\mathfrak{I},A}$ for $b \in [A]^n$. Further let Θ be the set of all $(f, g) \in (\mathfrak{I}^+)^{n+1} \times \kappa^{n+1}$ such that $f(n) \in \bigcup_{B \in Q_{ran(g \mid n)}} P(B)$. Then for every $C \in \mathfrak{I}^+ \cap P(A)$, I has a winning strategy in $G_{\mathfrak{I}}^*(0, C, \Theta)$.*

Proof. Use Proposition 3.1 to define for each $c \in [A]^{<n}$, $Q_c \in M_{\mathfrak{I},A}$, f_c: $Q_c \times A \to \mathfrak{I}^+$ and g_c: $Q_c \times A \to \mathfrak{I}$ so that

(i) for every $B \in Q_c$, $B \subseteq \{\alpha \in A\colon c \subseteq \alpha\}$;

(ii) given $B \in Q_c$ and $\alpha \in B$, $f_c(B, \alpha) \in Q_{c \cup \{\alpha\}}$ and $B - f_c(B, \alpha) = g_c(B, \alpha)$.

Given $C \in \mathfrak{I}^+ \cap P(A)$, we define a winning strategy σ for I in $G_{\mathfrak{I}}^*(0, C, \Theta)$ as follows. Let $A_0 \in Q_0$ be such that $A_0 \cap C \in \mathfrak{I}^+$, and set $\sigma(0) = A_0 \cap C$. Now let II successively play α_i for $i \leqslant n$. Define A_j and E_j for $0 < j \leqslant n$ so that $A_j = f_{\{\alpha_k \colon k < j-1\}}(A_{j-1}, \alpha_{j-1})$ and $E_j = g_{\{\alpha_k \colon k < j-1\}}(A_{j-1}, \alpha_{j-1})$. We set $\sigma(\alpha_0, \ldots, \alpha_m) = A_0 - \bigcup_{q \leqslant m} E_{q+1}$. $\square$

Proposition 9.2. *Assume either that $\mathfrak{I}$ is κ-distributive, or else that $\mathfrak{I}$ is almost κ-distributive and $add(\mathfrak{I}) > \aleph_0$. Let $A \in \mathfrak{I}^+$, and let $Q_b \in M_{\mathfrak{I},A}$ for $b \in [A]^{<\omega} - \{0\}$. Furthermore, let Θ be the set of all $(f, g) \in (\mathfrak{I}^+)^{\omega} \times \kappa^{\omega}$ such that for every $i \in \omega$, $f(i+1) \in \bigcup_{B \in Q_{\{g(j)\colon j \leqslant i\}}} P(B)$. Then for every $C \in \mathfrak{I}^+ \cap P(A)$, I has a winning strategy in $G_{\mathfrak{I}}^*(0, C, \Theta)$.*

Proof. Let us first assume that $\mathfrak{I}$ is κ-distributive. Pick $Q \in M_{\mathfrak{I},A}$ and $g : Q \to \prod_{b \in [A]^{<\omega} - \{0\}} Q_b$ such that for every $B \in Q$ and every $b \in [A]^{<\omega} - \{0\}$, $B - (g(B))(b) \in \mathfrak{I}$. Given $C \in \mathfrak{I}^+ \cap P(A)$, we define a winning strategy σ for I in $G_{\mathfrak{I}}^*(0, C, \Theta)$ as follows. Pick $A_0 \in Q$ with $A_0 \cap C \in \mathfrak{I}^+$. We set $\sigma(0) = A_0 \cap C$ and

$$\sigma(\alpha_0, \ldots, \alpha_i) = \left(A_0 \cap \bigcap_{k \leqslant i} (g(A_0))(\{\alpha_j \colon j \leqslant k\}) \right) - (\alpha_i + 1).$$

Let us now assume that $\mathfrak{I}$ is almost κ-distributive and $add(\mathfrak{I}) > \aleph_0$. By Lemma 9.1, there are $R_n \in M_{\mathfrak{I},A}$ and $\varphi_n : R_n \to \Sigma_{\mathfrak{I}}^{*\omega}$ for $n \in \omega - 1$ such that

(0) $(\varphi_n(B))(0) = B$;

(1) $\varphi_n(B)$ is a winning strategy for I in $G_{\mathfrak{I}}^*(0, A, \Theta_n)$, where Θ_n is the set of all $(f, g) \in (\mathfrak{I}^+)^{n+1} \times \kappa^{n+1}$ such that $f(n) \in \bigcup_{D \in Q_{ran(g \mid n)}} P(D)$.

By Proposition 3.3, there is an $R \in M_{\mathfrak{I},A}$ such that $R \leqslant R_n$ for all $n \in \omega - 1$. Given $C \in \mathfrak{I}^+ \cap P(A)$, we define a winning strategy σ for I in $G_{\mathfrak{I}}^*(0, C, \Theta)$ as follows. Pick

$E \in R$ with $E \cap C \in \mathfrak{I}^+$. Let $B_n \in R_n$ for $n \in \omega - 1$ be such that $E - B_n \in \mathfrak{I}$. We set

$$\sigma(0) = E \cap C \cap \bigcap_{n \in \omega - 1} B_n, \quad \sigma(\alpha_0) = \sigma(0) \cap \bigcap_{n \in \omega - 1} (\varphi_n(B_n))(\alpha_0)$$

and

$$\sigma(\alpha_0, \ldots, \alpha_{i+1}) = \sigma(\alpha_0, \ldots, \alpha_i) \cap \bigcap_{n \in \omega - (i+2)} (\varphi_n(B_n))(\alpha_0, \ldots, \alpha_{i+1}). \qquad \square$$

Let $\sigma \in \Sigma_{\mathfrak{I}}^{\omega}$. For every $b \in [\sigma(0)]^{<\omega} - \{0\}$, we define $Q_b^\sigma \in M_{\mathfrak{I}, \sigma(0)}$, $g_b^\sigma : Q_b^\sigma \to (\mathfrak{I}^+)^{|b|}$ and $t_b^\sigma : Q_b^\sigma \to (\mathfrak{I}^+)^{|b|}$ as follows.

Given $\alpha_0 \in \sigma(0)$, choose $Q_{\{\alpha_0\}}^\sigma \in M_{\mathfrak{I}, \sigma(0)}$ and

$$g_{\{\alpha_0\}}^\sigma : Q_{\{\alpha_0\}}^\sigma \to (\mathfrak{I}^+ \cap P(\sigma(0) - (\alpha_0 + 1)))^1$$

so that $A \subseteq (g_{\{\alpha_0\}}^\sigma(A))(0)$ and

$$Q_{\{\alpha_0\}}^\sigma = \{\sigma(\alpha_0, (g_{\{\alpha_0\}}^\sigma(A))(0)) : A \in Q_{\{\alpha_0\}}^\sigma\}.$$

Then define $t_{\{\alpha_0\}}^\sigma : Q_{\{\alpha_0\}}^\sigma \to (\mathfrak{I}^+)^1$ by letting $(t_{\{\alpha_0\}}^\sigma(A))(0) = \sigma(0)$.

Now let $n \in \omega$ and $b \in [\sigma(0)]^{n+2}$. Set $c = \{e_b(i) : i < n+1\}$. Given $A \in Q_c^\sigma$, we define Q_b^A and f_b^A as follows. Let us first assume that $e_b(n+1) \in A$ and for each $i \leqslant n$, $e_b(i) \in t_c^\sigma(A)(i)$. We choose $Q_b^A \in M_{\mathfrak{I}, A}$ and

$$f_b^A : Q_b^A \to \mathfrak{I}^+ \cap P(A - (e_b(n+1) + 1))$$

so that $D \subseteq f_b^A(D)$ and

$$Q_b^A = \{\sigma((e_b(0), (g_c^\sigma(A))(0)), \ldots, (e_b(n), (g_c^\sigma(A))(n)), (e_b(n+1), f_b^A(D))) : D \in Q_b^A\}.$$

Now suppose that $e_b(n+1) \notin A$, or that for some $i \leqslant n$, $e_b(i) \notin t_c^\sigma(A)(i)$. We put $Q_b^A = \{A\}$, and we define $f_b^A : Q_b^A \to \mathfrak{I}^+$ by letting $f_b^A(A) = A$. We next define $g_b^A : Q_b^A \to (\mathfrak{I}^+)^{n+2}$ and $t_b^A : Q_b^A \to (\mathfrak{I}^+)^{n+2}$ by letting

(0) $g_b^A(D) | n+1 = g_c^\sigma(A)$;
(1) $(g_b^A(D))(n+1) = f_b^A(D)$;
(2) $t_b^A(D) | n+1 = t_c^\sigma(A)$;
(3) $(t_b^A(A))(n+1) = A$.

We finally set $Q_b^\sigma = \bigcup_{A \in Q_c^\sigma} Q_b^A$, $g_b^\sigma = \bigcup_{A \in Q_c^\sigma} g_b^A$ and $t_b^\sigma = \bigcup_{A \in Q_c^\sigma} t_b^A$.

Assume either that $\mathfrak{I}$ is κ-distributive, or else that $\mathfrak{I}$ is almost κ-distributive and $add(\mathfrak{I}) > \aleph_0$. We define $\varepsilon_1 : \Sigma_{\mathfrak{I}}^{\omega} \to \Sigma_{\mathfrak{I}}^{*\omega}$ as follows. Let $\sigma \in \Sigma_{\mathfrak{I}}^{\omega}$. Define $\Theta \subseteq (\mathfrak{I}^+)^\omega \times \kappa^\omega$ by letting $(f, g) \in \Theta$ if and only if for every $i \in \omega$, $f(i+1) \in \bigcup_{B \in Q_{\{g(j) : j \leqslant i\}}^\sigma} P(B)$. Then using Proposition 9.2, we let $\varepsilon_1(\sigma)$ be a winning strategy for I in $G_{\mathfrak{I}}^*(0, \sigma(0), \Theta)$.

Notice that $\varepsilon_0(\varepsilon_1(\sigma)) \in \Phi(\sigma)$.

We will now deal with the following sequence of assumptions, that are increasing in strength:

(0) $\mathfrak{I}$ is either κ-distributive, or else almost κ-distributive and such that $add(\mathfrak{I}) > \aleph_0$.

(1) $\mathfrak{I}$ is κ-distributive and $add(\mathfrak{I}) = \kappa$.
(2) $\mathfrak{I}$ is a κ-distributive weak P-point.
(3) $\mathfrak{I}$ is κ-distributive and weakly semiselective.
(4) $\mathfrak{I}$ is κ-distributive and weakly selective.

To each stronger assumption will correspond a different, and in some sense better, characterization of $C_{\mathfrak{I}}$ (and of $N_{\mathfrak{I}}$). Let us observe the following, which concerns the case $\kappa = \omega$. Our list could be made longer by topping it off with the assumption (4′) that $\mathfrak{I}$ is weakly selective and $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$ is $\aleph_0$-closed, but the corresponding characterization of $C_{\mathfrak{I}}$, which was obtained by Mathias [24], is the same as that we derive from (4) (see Proposition 10.23).

Assumptions (0)–(4) will also be used to further our study of $add(N_{\mathfrak{I}})$ and other cardinals associated with $N_{\mathfrak{I}}$.

The remainder of this section is concerned with assumptions (0) and (1), whereas (2)–(4) will be dealt with in Section 10.

Proposition 9.3. *Assume either that $\mathfrak{I}$ is κ-distributive, or else that $\mathfrak{I}$ is almost κ-distributive and $add(\mathfrak{I}) > \aleph_0$. Then given $W \subseteq [\kappa]^\omega$, the following are equivalent:*

(i) $W \in N_{\mathfrak{I}}$.
(ii) *Given $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$, I has a winning strategy in $G^*_{\mathfrak{I}}(a, C, [\kappa]^\omega - W)$.*
(iii) *Given $a \in [\kappa]^{<\omega}$ and $\sigma \in \Sigma^{*\omega}_{\mathfrak{I}}$, there is a $\sigma' \in \Sigma^{*\omega}_{\mathfrak{I}}$ such that $\sigma' \leqslant \sigma$ and $\langle a, \sigma'\rangle^* \cap W = 0$.*
(iv) $\{\langle a,\sigma\rangle^* \in P^*_{\mathfrak{I}}: \langle a,\sigma\rangle^* \cap W = 0\}$ *is dense in* $(P^*_{\mathfrak{I}}, \subseteq)$.

Proof. (i) → (ii): Assume $W \in N_{\mathfrak{I}}$, and let $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$. Select a winning strategy σ for I in $G_{\mathfrak{I}}(a, C, [\kappa]^\omega - W)$. Then $\varepsilon_1(\sigma)$ is a winning strategy for I in $G^*_{\mathfrak{I}}(a, C, [\kappa]^\omega - W)$.

(ii) → (iii): Assume (ii), and let $a \in [\kappa]^{<\omega}$ and $\sigma \in \Sigma^{*\omega}_{\mathfrak{I}}$. Let τ be a winning strategy for I in $G^*_{\mathfrak{I}}(a, \sigma(0), [\kappa]^\omega - W)$. Then define $\sigma' \in \Sigma^{*\omega}_{\mathfrak{I}}$ by letting $\sigma'(0) = \tau(0)$ and $\sigma'(\alpha_0, \ldots, \alpha_i) = \sigma(\alpha_0, \ldots, \alpha_i) \cap \tau(\alpha_0, \ldots, \alpha_i)$.

(iii) → (iv): Trivial.

(iv) → (i): Assume (iv). Given $\langle a,\sigma\rangle \in P_{\mathfrak{I}}$, select $\langle b,\tau\rangle^* \in P^*_{\mathfrak{I}}$ such that $\langle b,\tau\rangle^* \subseteq \langle a, \varepsilon_1(\sigma)\rangle^*$ and $\langle b,\tau\rangle^* \cap W = 0$. It is readily checked that $\langle b, \varepsilon_0(\tau)\rangle \leqslant \langle a,\sigma\rangle$ and $\langle b, \varepsilon_0(\tau)\rangle \cap W = 0$. Thus the set $\{\langle a,\sigma\rangle \in P_{\mathfrak{I}}: \langle a,\sigma\rangle \cap W = 0\}$ is dense in $(P_{\mathfrak{I}}, \leqslant)$. Hence by Proposition 8.1, $W \in N_{\mathfrak{I}}$. □

Corollary 9.4. *Assume that $\mathfrak{I}$ is κ-distributive. Then $s_{\mathfrak{I}} \geqslant add(N_{\mathfrak{I}})$.*

Proof. Let $Z \subseteq M_{\mathfrak{I},\kappa}$ be such that $0 < |Z| < add(N_{\mathfrak{I}})$ and for every $Q \in Z$, $|Q| \leqslant 2$. For each $Q \in M_{\mathfrak{I},\kappa}$, set $W_Q = \{x \in [\kappa]^\omega: \exists B \in Q(x - B \in [\kappa]^{<\omega})\}$. Clearly, $[\kappa]^\omega - W_Q \in N_{\mathfrak{I}}$. Hence, $\bigcup_{Q \in Z}([\kappa]^\omega - W_Q) \in N_{\mathfrak{I}}$. Thus given $C \in \mathfrak{I}^+$, there is by Proposition 9.3 a winning strategy σ for I in $G^*_{\mathfrak{I}}(0, C, \bigcap_{Q \in Z} W_Q)$. We claim that for every $Q \in Z$, there is a $B \in Q$ such that $\sigma(0) - B \in \mathfrak{I}$. Suppose otherwise. Let $Q \in Z$ be such that $|Q| = 2$, and that for every $B \in Q$, $\sigma(0) \cap B \in \mathfrak{I}^+$. Put $Q = \{B_0, B_1\}$. Now define A_i and β_i for $i < \omega$ so that

(0) $A_0 = \sigma(0)$;

(1) $A_{k+1} = \sigma(\beta_0, \dots, \beta_k)$;

(2) $\beta_{2k} \in A_{2k} \cap B_0$;

(3) $\beta_{2k+1} \in A_{2k+1} \cap B_1$.

Then $\{\beta_i: i < \omega\} \notin \bigcap_{Q \in Z} W_Q$, a contradiction. □

For each $\sigma \in \Sigma_{\mathfrak{I}}^{*\omega}$, we define $F_\sigma : [\kappa]^{<\omega} - \{0\} \to \mathfrak{I}$ as follows. Let $n \in \omega$ and $\alpha_i \in \kappa$ for $i \leqslant n$ be such that $\alpha_j < \alpha_i$ whenever $j < i$. First assume that $\alpha_0 \in \sigma(0)$ and for each $j < n$, $\alpha_{j+1} \in \sigma(\alpha_0, \dots, \alpha_j)$. We put $F_\sigma(\{\alpha_i: i \leqslant n\}) = \sigma(0) - \sigma(\alpha_0, \dots, \alpha_n)$. Now assume otherwise. We set $F_\sigma(\{\alpha_i: i \leqslant n\}) = 0$.

Corollary 9.5. *Assume that $\mathfrak{I}$ is prime. Then $cof(\mathfrak{I}) \leqslant cof(N_{\mathfrak{I}})$.*

Proof. Let $X \subseteq N_{\mathfrak{I}}$ be such that $N_{\mathfrak{I}} = \bigcup_{W \in X} P(W)$. Given $W \in X$, there is by Proposition 9.3 a $\sigma_W \in \Sigma_{\mathfrak{I}}^{*\omega}$ such that $\langle 0, \sigma_W \rangle^* \cap W = 0$. We define $A_a^W \in \mathfrak{I}^*$ for $a \in [\kappa]^{<\omega}$ by letting $A_a^W = \sigma_W(0)$ in case $a = 0$, and $A_a^W = \sigma_W(0) - F_{\sigma_W}(a)$ otherwise. Now let $E \in \mathfrak{I}$. Select $W \in X$ such that $\{A \in [\kappa]^\omega: A \cap E \in [\kappa]^\omega\} \subseteq W$. It is readily seen that there exists $a \in [\kappa]^{<\omega}$ such that $E \cap A_a^W = 0$. □

Proposition 9.6. *Assume either that $\mathfrak{I}$ is κ-distributive, or else that $\mathfrak{I}$ is almost κ-distributive and $add(\mathfrak{I}) > \aleph_0$. Then given $W \subseteq [\kappa]^\omega$, the following are equivalent:*

(i) $W \in C_{\mathfrak{I}}$.

(ii) *Given $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$, I has either a winning strategy in $G_{\mathfrak{I}}^*(a, C, W)$, or else a winning strategy in $G_{\mathfrak{I}}^*(a, C, [\kappa]^\omega - W)$.*

(iii) *Given $a \in [\kappa]^{<\omega}$ and $\sigma \in \Sigma_{\mathfrak{I}}^{*\omega}$, there is a $\sigma' \in \Sigma_{\mathfrak{I}}^{*\omega}$ such that $\sigma' \leqslant \sigma$ and either $\langle a, \sigma' \rangle^* \subseteq W$, or else $\langle a, \sigma' \rangle^* \cap W = 0$.*

(iv) $\{\langle a, \sigma \rangle^* \in P_{\mathfrak{I}}^* : \langle a, \sigma \rangle^* \subseteq W$ or $\langle a, \sigma \rangle^* \cap W = 0\}$ *is dense in* $(P_{\mathfrak{I}}^*, \subseteq)$.

Proof. An easy modification of the proof of Proposition 9.3. □

Proposition 9.7 and Corollary 9.8 should be compared respectively with Proposition 8.5 and Corollary 8.6.

Proposition 9.7. *Assume $\mathfrak{I}$ is κ-distributive. Let μ be a cardinal with $0 < \mu < add(N_{\mathfrak{I}})$, and let $W_x \in C_{\mathfrak{I}}$ for $x \in \bigcup_{n \in \omega} \mu^n$. Then $\bigcup_{f \in \mu^\omega} \bigcap_{n \in \omega} W_{f|n} \in C_{\mathfrak{I}}$.*

Proof. For each $x \in \bigcup_{n \in \omega} \mu^n$, define T_x, Y_x and X_x as in the proof of Proposition 8.5. By the same proof, it is enough to show that each X_x is dense in $(P_{\mathfrak{I}}, \leqslant)$. Thus let $n \in \omega$, $x \in \mu^n$ and $\langle a, \sigma \rangle \in P_{\mathfrak{I}}$ be given. Let us first suppose that $\langle a, \varepsilon_1(\sigma) \rangle^* \cap T_x \in N_{\mathfrak{I}}$. Then by Proposition 9.3, there is a $\sigma' \in \Sigma_{\mathfrak{I}}^{*\omega}$ such that $\sigma' \leqslant \varepsilon_1(\sigma)$ and $\langle a, \sigma' \rangle^* \cap (\langle a, \varepsilon_1(\sigma) \rangle^* \cap T_x) = 0$. It is clear that $\langle a, \sigma' \rangle^* \cap T_x = 0$. Hence, $\langle a, \varepsilon_0(\sigma') \rangle \cap T_x = 0$. Moreover, as $\varepsilon_0(\sigma') \leqslant \varepsilon_0(\varepsilon_1(\sigma))$, we have $\langle a, \varepsilon_0(\sigma') \rangle \leqslant \langle a, \varepsilon_0(\varepsilon_1(\sigma)) \rangle \leqslant \langle a, \sigma \rangle$. Let us next suppose that there is a $\gamma \in \mu$ such that $\langle a, \varepsilon_1(\sigma)\rangle^* \cap T_{x \cup \{(n,\gamma)\}} \notin N_{\mathfrak{I}}$. Now by Proposition 9.3, there is a $\langle b_0, \tau_0\rangle^* \in P_{\mathfrak{I}}^*$ such that $\langle b, \tau\rangle^* \cap (\langle a, \varepsilon_1(\sigma)\rangle^* \cap T_{x \cup \{(n,\gamma)\}}) \neq 0$ whenever $\langle b, \tau\rangle^* \subseteq \langle b_0, \tau_0\rangle^*$. One then easily defines $\tau_1 \in \Sigma_{\mathfrak{I}}^{*\omega}$ such that $\tau_1 \leqslant \tau_0$ and $\langle b_0, \tau_1\rangle^* \subseteq \langle a, \varepsilon_1(\sigma)\rangle^*$. By Proposition 9.6, there is a $\tau_2 \in \Sigma_{\mathfrak{I}}^{*\omega}$ such that $\tau_2 \leqslant \tau_1$ and either $\langle b_0, \tau_2\rangle^* \subseteq W_x$, or else $\langle b_0, \tau_2\rangle^* \cap W_x = 0$. Clearly, $\langle b_0, \tau_2\rangle^* \subseteq W_x$ and, therefore, $\langle b_0, \varepsilon_0(\tau_2)\rangle \subseteq W_x$. It is readily checked that $\langle b_0, \varepsilon_0(\tau_2)\rangle \leqslant \langle b_0, \varepsilon_0(\tau_1)\rangle \leqslant \langle a, \sigma\rangle$. Notice that $\langle b, \tau\rangle^* \cap T_{x \cup \{(n,\gamma)\}} \neq 0$ whenever $\langle b, \tau\rangle^* \subseteq \langle b_0, \tau_2\rangle^*$. Thus given $\langle c, \rho\rangle \in P_{\mathfrak{I}}$ with $\langle c, \rho\rangle \leqslant \langle b_0, \varepsilon_0(\tau_2)\rangle$, we have $\langle c, \varepsilon_1(\rho)\rangle^* \subseteq \langle c, \rho\rangle \subseteq \langle b_0, \varepsilon_0(\tau_2)\rangle \subseteq \langle b_0, \tau_2\rangle^*$, and therefore $\langle c, \rho\rangle \cap T_{x \cup \{(n,\gamma)\}} \neq 0$. □

Corollary 9.8. *Assume $\mathfrak{I}$ is κ-distributive, $add(C_{\mathfrak{I}}) = add(N_{\mathfrak{I}})$.*

Proof. By Proposition 6.6 and Proposition 9.7. □

Proposition 9.9. *Assume that $\mathfrak{I}$ is κ-distributive and $add(\mathfrak{I}) = \kappa$. Then given $W \subseteq [\kappa]^\omega$, the following are equivalent:*

(i) $W \in N_{\mathfrak{I}}$.

(ii) *Given $\sigma \in \Sigma_{\mathfrak{I}}^{*\omega}$, there is a $\sigma' \in \Sigma_{\mathfrak{I}}^{*\omega}$ such that $\sigma' \leqslant \sigma$ and $\bigcup_{a \in [\kappa]^{<\omega}} \langle a, \sigma'\rangle^* \cap W = 0$.*

Proof. (i) → (ii): Assume $W \in N_{\mathfrak{I}}$, and let $\sigma \in \Sigma_{\mathfrak{I}}^{*\omega}$. By Proposition 8.2, there is a $\tau \in \Sigma_{\mathfrak{I}}^{\omega} | \varepsilon_0(\sigma)$ with $\bigcup_{a \in [\kappa]^{<\omega}} \langle a, \tau\rangle \cap W = 0$. Then $\varepsilon_1(\tau) \leqslant \sigma$ and $\bigcup_{a \in [\kappa]^{<\omega}} \langle a, \varepsilon_1(\tau)\rangle^* \cap W = 0$.

(ii) → (i): By Proposition 9.3. □

Given $\sigma \in \Sigma_{\mathfrak{I}}^{*\omega}$ and $\alpha \in \sigma(0)$, we define $\sigma | \alpha \in \Sigma_{\mathfrak{I}}^{*\omega}$ by letting $(\sigma | \alpha)(0) = \sigma(\alpha)$ and $(\sigma | \alpha)(\alpha_0, \ldots, \alpha_i) = \sigma(\alpha, \alpha_0, \ldots, \alpha_i)$.

Proposition 9.10. *Assume that $\mathfrak{I}$ is κ-distributive and $add(\mathfrak{I}) = \kappa$. Then given $W \subseteq [\kappa]^\omega$, the following are equivalent:*

(i) $W \in C_{\mathfrak{I}}$.

(ii) *Given $\sigma \in \Sigma_{\mathfrak{I}}^{*\omega}$, there is a $\sigma' \in \Sigma_{\mathfrak{I}}^{*\omega}$ such that $\sigma' \leqslant \sigma$ and for every $\alpha \in \sigma'(0)$ and every $a \in [\alpha + 1]^{<\omega}$, either $\langle a, \sigma' | \alpha\rangle^* \subseteq W$, or else $\langle a, \sigma' | \alpha\rangle^* \cap W = 0$.*

Proof. Use Propositions 8.4 and 9.6. □

10. Weak P-points

We set $\Sigma_{[\kappa]^{<\kappa}, \mathfrak{I}}^{*\omega} = \{\sigma \in \Sigma_{[\kappa]^{<\kappa}}^{*\omega} \colon \sigma(0) \in \mathfrak{I}^+\}$.

Assume that $\mathfrak{I}$ is a weak P-point. We define $\varepsilon_2 : \Sigma_{\mathfrak{I}}^{*\omega} \to \Sigma_{[\kappa]^{<\kappa}, \mathfrak{I}}^{*\omega}$ as follows. Let $\sigma \in \Sigma_{\mathfrak{I}}^{*\omega}$. By Proposition 1.7, there is an $A \in \mathfrak{I}^+ \cap P(\sigma(0))$ with the property that

$A - \sigma(\alpha_0, \ldots, \alpha_i) \in [\kappa]^{<\kappa}$. Then let $(\varepsilon_2(\sigma))(0) = A$ and $(\varepsilon_2(\sigma))(\alpha_0, \ldots, \alpha_i) = A \cap \sigma(\alpha_0, \ldots, \alpha_i)$.

It is clear that $\varepsilon_2(\sigma) \leqslant \sigma$.

We set $P^*_{[\kappa]^{<\kappa},\mathfrak{I}} = \{\langle a, \sigma\rangle^* \in P^*_{\mathfrak{I}}: a \in [\kappa]^{<\omega}$ and $\sigma \in \Sigma^{*\omega}_{[\kappa]^{<\kappa},\mathfrak{I}}\}$.

The following is straightforward.

Lemma 10.1. *Assume $\mathfrak{I}$ is a weak P-point. Then $P^{*\omega}_{[\kappa]^{<\kappa},\mathfrak{I}}$ is dense in $(P^*_{\mathfrak{I}}, \subseteq)$.*

Given $D \in \mathfrak{I}^+$ and an increasing $f \in \kappa^\kappa$, we define $\sigma_{D,f} \in \Sigma^{*\omega}_{[\kappa]^{<\kappa},\mathfrak{I}}$ by letting $\sigma_{D,f}(0) = D$ and $\sigma_{D,f}(\alpha_0, \ldots, \alpha_i) = D - (f(\alpha_i)) \cup (\alpha_i + 1))$.

Lemma 10.2. *Assume $\mathfrak{I}$ is a weak P-point, and let $\sigma \in \Sigma^{*\omega}_{[\kappa]^{<\kappa},\mathfrak{I}}$. Then there are $D \in \mathfrak{I}^+$ and an increasing $f \in \kappa^\kappa$ such that $\sigma_{D,f} \leqslant \sigma$.*

Proof. Define $H: [\kappa]^{<\omega} - \{0\} \to \kappa$ by letting $H(b) = (\bigcup F_\sigma(b)) + 1$. Then define $f \in \kappa^\kappa$ by letting $f(\beta) = \bigcup \{H(b): b \in [\beta + 1]^{<\omega} - \{0\}\}$. Setting $D = \sigma(0)$, we clearly have that $\sigma_{D,f} \leqslant \sigma$. □

Proposition 10.3. *Assume that $\mathfrak{I}$ is a κ-distributive weak P-point. Then given $W \subseteq [\kappa]^\omega$, the following are equivalent:*

(i) $W \in N_{\mathfrak{I}}$.

(ii) *Given $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$, there is a $\sigma \in \Sigma^{*\omega}_{[\kappa]^{<\kappa},\mathfrak{I}}$ such that $\sigma(0) \subseteq C$ and $\langle a, \sigma\rangle^* \cap W = 0$.*

(iii) *Given $a \in [\kappa]^{<\omega}$ and $\sigma \in \Sigma^{*\omega}_{[\kappa]^{<\kappa},\mathfrak{I}}$, there is a $\sigma' \in \Sigma^{*\omega}_{[\kappa]^{<\kappa},\mathfrak{I}}$ such that $\sigma' \leqslant \sigma$ and $\langle a, \sigma'\rangle^* \cap W = 0$.*

(iv) *Given $\sigma \in \Sigma^{*\omega}_{[\kappa]^{<\kappa},\mathfrak{I}}$, there is a $\sigma' \in \Sigma^{*\omega}_{[\kappa]^{<\kappa},\mathfrak{I}}$ such that $\sigma' \leqslant \sigma$ and $\bigcup_{a \in [\kappa]^{<\omega}} \langle a, \sigma'\rangle^* \cap W = 0$.*

(v) *Given $C \in \mathfrak{I}^+$, there are $D \in \mathfrak{I}^+ \cap P(C)$ and an increasing $f \in \kappa^\kappa$ such that $\bigcup_{a \in [\kappa]^{<\omega}} \langle a, \sigma_{D,f}\rangle^* \cap W = 0$.*

(vi) *$\{\langle a, \sigma\rangle^* \in P^*_{[\kappa]^{<\kappa},\mathfrak{I}}: \langle a, \sigma\rangle^* \cap W = 0\}$ is dense in $(P^*_{[\kappa]^{<\kappa},\mathfrak{I}}, \subseteq)$.*

Proof. (i) → (iv): Assume $W \in N_{\mathfrak{I}}$, and let $\sigma \in \Sigma^{*\omega}_{[\kappa]^{<\kappa},\mathfrak{I}}$. By Proposition 9.9, there is a $\sigma' \in \Sigma^{*\omega}_{\mathfrak{I}}$ such that $\sigma' \leqslant \sigma$ and $\bigcup_{a \in [\kappa]^{<\omega}} \langle a, \sigma'\rangle^* \cap W = 0$. Then $\varepsilon_2(\sigma') \leqslant \sigma$ and $\bigcup_{a \in [\kappa]^{<\omega}} \langle a, \varepsilon_2(\sigma')\rangle^* \cap W = 0$.

(iv) → (v) → (ii) → (iii) → (vi): Left to the reader.

(vi) → (i): By Lemma 10.1 and Proposition 9.3. □

We will now resume our study of the cardinals associated with $N_{\mathfrak{I}}$. Assuming $\mathfrak{I}$ is a κ-distributive weak P-point, we are able to compute $cov(N_{\mathfrak{I}})$ and $non(N_{\mathfrak{I}})$ (see Section 5 and Propositions 10.6 and 10.7). The computation of $add(N_{\mathfrak{I}})$ and $cof(N_{\mathfrak{I}})$ is more difficult and will be accomplished under the stronger assumption that $\mathfrak{I}$ is a prime weak P-point or is κ-distributive and weakly selective (see Section 5, Corollary 10.5, Proposition 10.10, Proposition 10.21 and its proof).

Proposition 10.4. *Assume that $\kappa = \omega$ and $\mathfrak{I}$ is a κ-distributive weak P-point. Then $\boldsymbol{b} \cap \boldsymbol{k}_{\mathfrak{I}} \leqslant add(N_{\mathfrak{I}}) \leqslant \boldsymbol{b} \cap \pi_{\mathfrak{I}}$.*

Proof. By Proposition 5.5, $add(N_{\mathfrak{I}}) \leqslant \boldsymbol{b}$. Now let $X \subseteq \mathfrak{I}$ with $0 < |X| < add(N_{\mathfrak{I}})$. For each $B \in X$, set $W_B = \{E \in [\omega]^{\omega}: E \cap B \in [\omega]^{\omega}\}$. Clearly, $W_B \in N_{\mathfrak{I}}$ for every $B \in X$, and therefore $\bigcup_{B \in X} W_B \in N_{\mathfrak{I}}$. Given $C \in \mathfrak{I}^+$, there is by Proposition 10.3 a $\tau \in \Sigma^{*\omega}_{[\omega]^{<\omega},\mathfrak{I}}$ such that $\tau(0) \subseteq C$ and $\langle 0, \tau\rangle^* \cap \bigcup_{B \in X} W_B = 0$. It is immediate that $\tau(0) \cap B \in [\omega]^{<\omega}$ for every $B \in X$. Thus $add(N_{\mathfrak{I}}) \leqslant \pi_{\mathfrak{I}}$.

It remains to show that $add(N_{\mathfrak{I}}) \geqslant \boldsymbol{b} \cap \boldsymbol{k}_{\mathfrak{I}}$. Thus let $Z \subseteq N_{\mathfrak{I}}$ with $0 < |Z| < \boldsymbol{b} \cap \boldsymbol{k}_{\mathfrak{I}}$. For each $W \in Z$, there are by Proposition 10.3 $Q_W \in M_{\mathfrak{I},\omega}$ and $F_W: Q_W \to \omega^{\omega}$ such that for every $B \in Q_W$, $F_W(B)$ is increasing and $\bigcup_{a \in [\omega]^{<\omega}} \langle a, \sigma_{B,F_W(B)}\rangle^* \cap W = 0$. Now fix $a \in [\omega]^{<\omega}$ and $C \in \mathfrak{I}^+$. Select $D \in \mathfrak{I}^+ \cap P(C)$ and $g \in \prod_{W \in Z} Q_W$ so that for every $W \in Z$, $D - g(W) \in [\omega]^{<\omega}$. Then let $f \in \omega^{\omega}$ be strictly increasing and such that for every $W \in Z$, $\{n: f(n) \leqslant (F_W(g(W)))(n)\} \in [\omega]^{<\omega}$. Given $W \in Z$, pick $m_W \in \omega$ so that $D - m_W \subseteq g(W)$ and for every $n \geqslant m_W$, $f(n) > (F_W(g(W)))(n)$. For each $E \in \langle a, \sigma_{D,f}\rangle^*$, we have $E \in \langle a \cup (E \cap m_W), \sigma_{g(W),F_W(g(W))}\rangle^*$. Thus $\langle a, \sigma_{D,f}\rangle^* \cap W = 0$. Hence by Proposition 10.3 $\bigcup Z \in N_{\mathfrak{I}}$. $\square$

Corollary 10.5. *Assume that $\kappa = \omega$ and $\mathfrak{I}$ is a prime weak P-point. Then $add\,(N_{\mathfrak{I}}) = \boldsymbol{b} \cap \boldsymbol{k}_{\mathfrak{I}}$.*

Proof. By Proposition 10.4 and Proposition 2.7. $\square$

Proposition 10.6. *Assume that $\kappa = \omega$ and $\mathfrak{I}$ is a κ-distributive weak P-point. Then $cov(N_{\mathfrak{I}}) = \boldsymbol{b} \cap \chi_{\mathfrak{I}}$.*

Proof. We have that $cov(N_{\mathfrak{I}}) \leqslant \boldsymbol{b} \cap \chi_{\mathfrak{I}}$ by the results of Section 5. Let us show that $cov(N_{\mathfrak{I}}) \geqslant \boldsymbol{b} \cap \chi_{\mathfrak{I}}$. Thus, fix $Z \subseteq N_{\mathfrak{I}}$ with $0 < |Z| < \boldsymbol{b} \cap \chi_{\mathfrak{I}}$. For each $W \in Z$, there are by Proposition 10.3 $Q_W \in M_{\mathfrak{I},\omega}$ and $F_W: Q_W \to \omega^{\omega}$ such that for every $B \in Q_W$, $F_W(B)$ is increasing and $\bigcup_{a \in [\omega]^{<\omega}} \langle a, \sigma_{B,F_W(B)}\rangle^* \cap W = 0$. Select $E \in [\omega]^{\omega}$ and $g \in \prod_{W \in Z} Q_W$ so that for every $W \in Z$, $E - g(W) \in [\omega]^{<\omega}$. Then let $f \in \omega^{\omega}$ be strictly increasing and such that for every $W \in Z$, $\{n: f(n) \leqslant (F_W(g(W)))(n)\} \in [\omega]^{<\omega}$. Define a strictly increasing $h \in E^{\omega}$ so that for each $n, h(n+1) > f(h(n))$. Given $W \in Z$, pick $m_W \in \omega$ so that $E - m_W \subseteq g(W)$ and for every $n \geqslant m_W$, $f(n) > (F_W(g(W)))(n)$. Then $ran(h) \in \langle m_W \cap ran(h), \sigma_{g(W),F_W(g(W))}\rangle^*$. $\square$

Proposition 10.7. *Assume that $\kappa = \omega$ and $\mathfrak{I}$ is a κ-distributive weak P-point. Then $non(N_{\mathfrak{I}}) = \boldsymbol{d} \cup \nu_{\mathfrak{I}}$.*

Proof. We have that $non(N_{\mathfrak{I}}) \geqslant \boldsymbol{d} \cup \nu_{\mathfrak{I}}$ by Proposition 5.10 and Proposition 5.11. Let us show that $non(N_{\mathfrak{I}}) \leqslant \boldsymbol{d} \cup \nu_{\mathfrak{I}}$. Let $Z \subseteq [\omega]^{\omega}$ be such that for every $Q \in M_{\mathfrak{I},\omega}$, $\{A - B: A \in W \text{ and } B \in Q\} \cap [\omega]^{<\omega} \neq 0$, and let $Y \subseteq \omega^{\omega}$ be such that for every $f \in \omega^{\omega}$, there is a $g \in Y$ with $\{n: f(n) > g(n)\} \in [\omega]^{<\omega}$. Given $A \in Z$ and $g \in Y$, select a strictly increasing $h_{A,g} \in A^\omega$ so that for every $n \in \omega$, $h_{A,g}(n+1) \geqslant g(h_{A,g}(n))$. Given $W \in N_\mathfrak{I}$, there are by Proposition 10.3 $Q \in M_{\mathfrak{I},\omega}$ and $F: Q \to \omega^\omega$ such that for every $B \in Q$, $F(B)$ is increasing and $\bigcup_{a \in [\omega]^{<\omega}} \langle a, \sigma_{B,F(B)} \rangle^* \cap W = 0$. Now pick $A \in Z$ and $B \in Q$ so that $A - B \in [\omega]^{<\omega}$. Select $g \in Y$ so that $\{n\colon (F(B))(n) > g(n)\} \in [\omega]^{<\omega}$. Let $m \in \omega$ be such that $A - m \subseteq B$ and for every $n \geqslant m$, $g(n) \geqslant (F(B))(n)$. Then $ran(h_{A,g}) \in \langle m \cap ran(h_{A,g}), \sigma_{B,F(B)} \rangle^*$. Hence, $ran(h_{A,g}) \notin W$. □

Proposition 10.8. *Assume that $\mathfrak{I}$ is a κ-distributive weak P-point. Then $\boldsymbol{d}_\kappa \leqslant cof(N_\mathfrak{I})$.*

Proof. The result is immediate from Proposition 5.11 in case $\kappa = \omega$. So let us assume that $\kappa > \omega$. Let $Z \subseteq N_\mathfrak{I}$ be such that $0 < |Z| < \boldsymbol{d}_\kappa$. Given $W \in Z$, there is by Proposition 10.3 a $\sigma_W \in \Sigma^{*\omega}_{[\kappa]^{<\kappa},\mathfrak{I}}$ such that $W \cap \langle 0, \sigma_W \rangle^* = 0$. Define $f_W: \kappa \times \omega \to \kappa$ by letting $f_W(\alpha, 0) = \bigcap (\sigma_W(0) - \alpha)$ and

$$f_W(\alpha, n+1) = \bigcap \sigma_W(f_W(\alpha,0), \ldots, f_W(\alpha,n)).$$

Then define $k_W \in \kappa^\kappa$ by letting $k_W(\alpha) = \bigcup_{n \in \omega} f_W(\alpha, n)$. Let $g \in \kappa^\kappa$ be strictly increasing and such that for every $W \in Z$, $\{\alpha\colon g(\alpha) \geqslant k_W(\alpha)\} \neq 0$. Set $X = [\kappa]^\omega - \bigcup_{a \in [\kappa]^{<\omega}} \langle a, \sigma_{\kappa,g} \rangle^*$. By Proposition 10.3, $X \in N_\mathfrak{I}$. Given $W \in Z$, pick $\alpha \in \kappa$ with $g(\alpha) \geqslant k_W(\alpha)$. For every $n \in \omega$, we have

$$\bigcap \sigma_{\kappa,g}(f_W(\alpha,n)) \geqslant g(f_W(\alpha,n)) \geqslant g(\alpha) > f_W(\alpha, n+1)$$

and therefore $\{f_W(\alpha,i)\colon i \in \omega\} \notin \langle \{f_W(\alpha,j)\colon j < n\}, \sigma_{\kappa,g} \rangle^*$. Thus $\{f_W(\alpha,i)\colon i \in \omega\} \in \langle 0, \sigma_W \rangle^* - \bigcup_{a \in [\kappa]^{<\omega}} \langle a, \sigma_{\kappa,g} \rangle^*$. Hence $X - W \neq 0$. □

The following is straightforward.

Lemma 10.9. *Assume that $\mathfrak{I}$ is a prime weak P-point. Then given $W \subseteq [\kappa]^\omega$, $W \in N_\mathfrak{I}$ if and only if there are $D \in \mathfrak{I}^*$ and an increasing $f \in \kappa^\kappa$ with $W \cap \bigcup_{a \in [\kappa]^{<\omega}} \langle a, \sigma_{D,f} \rangle^* = 0$.*

Proposition 10.10. *Assume that $\mathfrak{I}$ is a prime weak P-point. Then $cof(N_\mathfrak{I}) = \boldsymbol{d}_\kappa \cup cof(\mathfrak{I})$.*

Proof. We have $cof(N_\mathfrak{I}) \geqslant \boldsymbol{d}_\kappa \cup cof(\mathfrak{I})$ by Corollary 9.5 and Proposition 10.8. Let us show that $cof(N_\mathfrak{I}) \leqslant \boldsymbol{d}_\kappa \cup cof(\mathfrak{I})$. Let $Y \subseteq \kappa^\kappa$ be such that for every $f \in \kappa^\kappa$, there is a $g \in Y$ with $\{\alpha: f(\alpha) > g(\alpha)\} \in [\kappa]^{<\kappa}$. Let $K \subseteq \mathfrak{I}$ be such that $\mathfrak{I} = \bigcup_{B \in K} P(B)$. Then given $W \in N_\mathfrak{I}$, there are by Lemma 10.9 $g \in Y$, $B \in K$ and $\alpha < \kappa$ such that $W \cap \bigcup_{a \in [\kappa]^{<\omega}} \langle a, \sigma_{\kappa - (B \cup \alpha), g} \rangle^* = 0$. We have $[\kappa]^\omega - \bigcup_{a \in [\kappa]^{<\omega}} \langle a, \sigma_{\kappa - (B \cup \alpha), g} \rangle^* \in N_\mathfrak{I}$ by Lemma 10.9. □

Proposition 10.11. *Assume that $\mathfrak{I}$ is a κ-distributive weak P-point. Then given $W \subseteq [\kappa]^\omega$, the following are equivalent*:

(i) $W \in C_\mathfrak{I}$.

(ii) *Given $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$, there is a $\sigma \in \Sigma^{*\omega}_{[\kappa]^{<\kappa},\mathfrak{I}}$ such that $\sigma(0) \subseteq C$ and either $\langle a, \sigma' \rangle^* \subseteq W$, or else $\langle a, \sigma' \rangle^* \cap W = 0$.*

(iii) *Given* $a \in [\kappa]^{<\omega}$ *and* $C \in \mathfrak{I}^+$, *there are* $D \in \mathfrak{I}^+ \cap P(C)$ *and an increasing* $f \in \kappa^\kappa$ *such that either* $\langle a, \sigma_{D,f}\rangle^* \subseteq W$, *or else* $\langle a, \sigma_{D,f}\rangle^* \cap W = 0$.

(iv) *Given* $\sigma \in \Sigma^{*\omega}_{[\kappa]^{<\kappa},\mathfrak{I}}$, *there is a* $\sigma' \in \Sigma^{*\omega}_{[\kappa]^{<\kappa},\mathfrak{I}}$ *such that* $\sigma' \leqslant \sigma$ *and for every* $\alpha \in \sigma(0)$ *and every* $a \in [\alpha+1]^{<\omega}$, *either* $\langle a, \sigma' | \alpha\rangle^* \subseteq W$, *or else* $\langle a, \sigma' | \alpha\rangle^* \cap W = 0$.

(v) $\{\langle a,\sigma\rangle^* \in P^*_{[\kappa]^{<\kappa},\mathfrak{I}} : \langle a,\sigma\rangle^* \subseteq W$ *or* $\langle a,\sigma\rangle^* \cap W = 0\}$ *is dense in* $(P^*_{[\kappa]^{<\kappa},\mathfrak{I}}, \subseteq)$.

Proof. Left to the reader. □

Given $A \in \mathfrak{I}^+$, we define $\sigma_A \in \Sigma^{*\omega}_{[\kappa]^{<\kappa},\mathfrak{I}}$ by letting $\sigma_A(0) = A$ and $\sigma_A(\alpha_0, \ldots, \alpha_i) = A - e_A(\alpha_i + 1)$.

Lemma 10.12. *Assume* $\mathfrak{I}$ *is a weak semi-Q-point. Then given* $D \in \mathfrak{I}^+$ *and an increasing* $f \in \kappa^\kappa$, *there is an* $A \in \mathfrak{I}^+$ *such that* $\sigma_A \leqslant \sigma_{D,f}$.

Proof. Select $A \in \mathfrak{I}^+ \cap P(D)$ so that for every $\alpha \in \kappa$, $e_A(\alpha+1) \geqslant f(\alpha)$. □

Proposition 10.13. *Assume that* $\mathfrak{I}$ *is* κ*-distributive and weakly semiselective. Then given* $W \subseteq [\kappa]^\omega$, $W \in N_\mathfrak{I}$ *if and only if given* $C \in \mathfrak{I}^+$, *there is an* $A \in \mathfrak{I}^+ \cap P(C)$ *such that* $\bigcup_{a \in [\kappa]^{<\omega}} \langle a, \sigma_A\rangle^* \cap W = 0$.

Proof. By Proposition 10.3 and Lemma 10.12. □

Proposition 10.14. *Assume that* $\kappa = \omega$ *and* $\mathfrak{I}$ *is prime and weakly semiselective. Then* $add(N_\mathfrak{I}) = \boldsymbol{k}_\mathfrak{I}$.

Proof. By Corollary 10.5 and Proposition 2.9. □

Proposition 10.15. *Assume that* $\kappa = \omega$ *and* $\mathfrak{I}$ *is* κ*-distributive and weakly semiselective. Then* $cov(N_\mathfrak{I}) = \chi_\mathfrak{I}$ *and* $non(N_\mathfrak{I}) = \nu_\mathfrak{I}$.

Proof. By Propositions 10.6, 2.9, 10.7 and 1.11. □

Proposition 10.16. *Assume that* $\mathfrak{I}$ *is prime and weakly semiselective. Then* $cof(N_\mathfrak{I}) = cof(\mathfrak{I})$.

Proof. By Proposition 10.10 and the following easy fact: Assume that $\mathfrak{I}$ is a prime weak semi-Q-point. Then $cof(\mathfrak{I}) \geqslant \boldsymbol{d}_\kappa$. □

Proposition 10.17. *Assume that* $\mathfrak{I}$ *is* κ*-distributive and weakly semiselective. Then* $cof(N_\mathfrak{I}) \leqslant m_\mathfrak{I}$.

Proof. Let $X \subseteq M_{\mathfrak{I},\kappa}$ be such that for every $Q \in M_{\mathfrak{I},\kappa}$, there is an $R \in X$ with $R \subseteq \{B \cup c\colon B \in Q$ and $c \in [\kappa]^{<\omega}\}$. For each $A \in [\kappa]^\kappa$, define $f_A \in \kappa^\kappa$ by letting

$f_A(\alpha) = e_A(2\alpha + 1)$ in case $\kappa = \omega$, and $f_A(\alpha) = e_A(\alpha + \omega)$ otherwise. Given $R \in X$, set $W_R = [\kappa]^\omega - \bigcup_{A \in R} \bigcup_{a \in [\kappa]^{<\omega}} \langle a, \sigma_{A,f_A} \rangle^*$. It is readily checked that $W_R \in N_{\mathfrak{I}}$. Now fix $W \in N_{\mathfrak{I}}$. By Proposition 10.13 there is a $Q \in M_{\mathfrak{I},\kappa}$ such that $W \cap \bigcup_{B \in Q} \bigcup_{b \in [\kappa]^{<\omega}} \langle b, \sigma_B \rangle^* = 0$. Select $R \in X$ so that $R \subseteq \{B \cup c\colon B \in Q$ and $c \in [\kappa]^{<\omega}\}$. Given $A \in R$, pick $B \in Q$ so that $A - B \in [\kappa]^{<\omega}$. Setting $E = \{\alpha \in \kappa\colon f_A(\alpha) < e_B(\alpha + 1)\}$, it is easy to see that $E \in [\kappa]^{<\omega}$ in case $\kappa = \omega$, and $E = 0$ otherwise. Hence for each $a \in [\kappa]^{<\omega}$, $\langle a, \sigma_{A,f_A} \rangle^* \subseteq \bigcup_{b \in [\kappa]^{<\omega}} \langle b, \sigma_B \rangle^*$. Thus, $W \subseteq W_R$. □

Proposition 10.18. *Assume that $\mathfrak{I}$ is κ-distributive and weakly semiselective. Then given $W \subseteq [\kappa]^\omega$, $W \in C_{\mathfrak{I}}$ if and only if given $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$, there is an $A \in \mathfrak{I}^+ \cap P(C)$ such that either $\langle a, \sigma_A \rangle^* \subseteq W$, or else $\langle a, \sigma_A \rangle^* \cap W = 0$.*

Proof. By Proposition 10.11 and Lemma 10.12. □

Given $C \in \mathfrak{I}^+$, we define $\lfloor C \rfloor \in \Sigma_{\mathfrak{I}}^{*\omega}$ by letting $\lfloor C \rfloor(0) = C$ and $\lfloor C \rfloor(\alpha_0, \ldots, \alpha_i) = C - (\alpha_i + 1)$.

For each $a \in [\kappa]^{<\omega}$, we let $\langle a, C \rangle$ be the set of all $D \in [\kappa]^\omega$ with $a \subset D \subseteq a \cup \{\alpha \in C\colon a \subseteq \alpha\}$.

Notice that $\langle a, C \rangle = \langle a, \lceil C \rceil \rangle = \langle a, \lfloor C \rfloor \rangle^*$.

We set $Q_{\mathfrak{I}} = \{\langle a, C \rangle\colon a \in [\kappa]^{<\omega}$ *and* $C \in \mathfrak{I}^+\}$.

Assume that $\mathfrak{I}$ is a weak Q-point. We define $\varepsilon_3 : \Sigma^{*\omega}_{[\kappa]^{<\kappa},\mathfrak{I}} \to \mathfrak{I}^+$ as follows. Let $\sigma \in \Sigma^{*\omega}_{[\kappa]^{<\kappa},\mathfrak{I}}$. For each $\gamma \in \sigma(0)$, we define $H_\gamma \subseteq \kappa$ by letting $\beta \in H_\gamma$ if and only if there are $i < \omega$ and $\alpha_j \leqslant \gamma$ for $j \leqslant i$ such that $\beta \in \sigma(0) - \sigma(\alpha_0, \ldots, \alpha_i)$. By Lemma 1.8, there is an $A \in \mathfrak{I}^+ \cap P(\sigma(0))$ such that $\beta \notin \bigcup_{\gamma \in A \cap \beta} H_\gamma$ for all $\beta \in A$. Now put $\varepsilon_3(\sigma) = A$.

It is clear that for every $a \in [\kappa]^{<\omega}$, $\langle a, \varepsilon_3(\sigma) \rangle \subseteq \langle a, \sigma \rangle^*$. The following is straightforward.

Lemma 10.19. *Assume $\mathfrak{I}$ is a weak Q-point. Then $\{\langle a, \lfloor C \rfloor \rangle^*\colon a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+\}$ is dense in $(P^*_{[\kappa]^{<\kappa},\mathfrak{I}}, \subseteq)$.*

Proposition 10.20. *Assume that $\mathfrak{I}$ is κ-distributive and weakly selective. Then given $W \subseteq [\kappa]^\omega$, the following are equivalent:*

(i) $W \in N_{\mathfrak{I}}$.

(ii) *Given $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$, there is a $D \in \mathfrak{I}^+ \cap P(C)$ with $\langle a, D \rangle \cap W = 0$.*

(iii) *Given $C \in \mathfrak{I}^+$, there is a $D \in \mathfrak{I}^+ \cap P(C)$ with $\bigcup_{a \in [\kappa]^{<\omega}} \langle a, D \rangle \cap W = 0$.*

(iv) *$\{\langle a, C \rangle \in Q_{\mathfrak{I}}\colon \langle a, C \rangle \cap W = 0\}$ is dense in $(Q_{\mathfrak{I}}, \subseteq)$.*

Proof. Use Proposition 10.3 and Lemma 10.19. □

Proposition 10.21. *Assume that $\kappa = \omega$ and $\mathfrak{I}$ is κ-distributive and weakly selective. Then $add(N_{\mathfrak{I}}) = \boldsymbol{k}_{\mathfrak{I}}$ and $cof(N_{\mathfrak{I}}) = m_{\mathfrak{I}}$.*

Proof. Let us first show that $add(N_{\mathfrak{I}}) = \boldsymbol{k}_{\mathfrak{I}}$. By Propositions 10.4, 2.7 and 2.9, it suffices to show that $add(N_{\mathfrak{I}}) \leqslant \boldsymbol{h}_{\mathfrak{I}}$. Thus let $X \subseteq M_{\mathfrak{I},\omega}$ and $E \in \mathfrak{I}^+$ be such that for every $D \in \mathfrak{I}^+ \cap P(E)$, there is a $Q \in X$ with $\{D - B\colon B \in Q\} \subseteq \mathfrak{I}^+$. For each $Q \in M_{\mathfrak{I},\omega}$, set

$$W_Q = \{x \in [\omega]^\omega\colon \exists B \in Q(x - B \in [\kappa]^{<\omega})\}.$$

Clearly, $[\omega]^\omega - W_Q \in N_{\mathfrak{I}}$. We have $P(E) \cap \bigcap_{Q \in X} W_Q \subseteq \mathfrak{I}$. Hence by Proposition 10.20, $\bigcup_{Q \in X}([\omega]^\omega - W_Q) \notin N_{\mathfrak{I}}$.

Let us next show that $cof(N_{\mathfrak{I}}) = \boldsymbol{m}_{\mathfrak{I}}$. The result easily follows from the following remark. Given $W \subseteq [\kappa]^\omega$, we have by Proposition 10.20 that $W \in N_{\mathfrak{I}}$ if and only if there is a $Q \in M_{\mathfrak{I},\kappa}$ such that $W \cap \bigcup_{B \in Q} \bigcup_{a \in [\kappa]^{<\omega}} \langle a, B\rangle = 0$. □

Corollary 10.22. *Assume that $\kappa = \omega$ and $\mathfrak{I}$ is feeble, κ-distributive and weakly selective. Then $cof(N_{\mathfrak{I}}) > 2^{\aleph_0}$.*

Proof. By Propositions 1.3, 2.11 and 10.21. □

Proposition 10.23. *Assume that $\mathfrak{I}$ is κ-distributive and weakly selective. Then given $W \subseteq [\kappa]^\omega$, the following are equivalent:*

(i) $W \in C_{\mathfrak{I}}$.

(ii) *Given $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$, there is a $D \in \mathfrak{I}^+ \cap P(C)$ such that either $\langle a, D\rangle \subseteq W$, or else $\langle a, D\rangle \cap W = 0$.*

(iii) *Given $C \in \mathfrak{I}^+$, there is a $D \in \mathfrak{I}^+ \cap P(C)$ such that for every $\alpha \in D$ and every $a \in [\alpha + 1]^{<\omega}$, either $\langle a, D - (\alpha + 1)\rangle \subseteq W$, or else $\langle a, D - (\alpha + 1)\rangle \cap W = 0$.*

(iv) *$\{\langle a, C\rangle \in Q_{\mathfrak{I}}\colon \langle a, C\rangle \subseteq W$ or $\langle a, C\rangle \cap W = 0\}$ is dense in $(Q_{\mathfrak{I}}, \subseteq)$.*

Proof. Left to the reader. □

11. $P_{\mathfrak{I}}$-generic subsets of κ

In the next three sections we will study a notion of forcing that can be seen as a generalization of Mathias forcing (and of Prikry forcing). The definition will again involve strategies, but we will see in Section 13 that $(P_{\mathfrak{I}}, \leqslant)$ and (the more familiar) $(Q_{\mathfrak{I}}, \subseteq)$ yield the same generic extensions in case $\mathfrak{I}$ is κ-distributive and weakly selective.

It is customary to formulate results directly in terms of 'Mathias reals' or 'Prikry sequences', thus bypassing the generic sets those objects were defined from. We will conform to that convenient usage.

We say that $x \in [\kappa]^\omega$ is *$P_{\mathfrak{I}}$-generic over V* if whenever $X \in P(P_{\mathfrak{I}})$ is dense in $(P_{\mathfrak{I}}, \leqslant)$, there exists $\langle a, \sigma\rangle \in X$ with $x \in \langle a, \sigma\rangle$.

We start by mentioning some elementary properties of $P_{\mathfrak{I}}$-generic sets. Note that the results of this section are valid for arbitrary ideals.

Let $x \in [\kappa]^\omega$ be $P_{\mathfrak{I}}$-generic over V. We set $G_x = \{\langle a, \sigma\rangle \in P_{\mathfrak{I}}\colon x \in \langle a, \sigma\rangle\}$.

Proposition 11.1. *Given $x \in [\kappa]^\omega$, the following are equivalent:*

(i) *x is P_3-generic over V.*

(ii) *Whenever $X \in P(P_3)$ is dense and open in $(P_3, \leqslant)$, there exists $\langle a, \sigma \rangle \in X$ with $x \in \langle a, \sigma \rangle$.*

Proof. (i) $\to$ (ii): Trivial.

(ii) $\to$ (i): Assume (ii), and let $X \subseteq P_3$ be dense in $(P_3, \leqslant)$. Let Y be the set of all $\langle a', \sigma' \rangle \in P_3$ such that there is an $\langle a, \sigma \rangle \in X$ with $\langle a', \sigma' \rangle \leqslant \langle a, \sigma \rangle$. Y is clearly dense and open in $(P_3, \leqslant)$. Pick $\langle a', \sigma' \rangle \in Y \cap G_x$, and let $\langle a, \sigma \rangle \in X$ be such that $\langle a', \sigma' \rangle \leqslant \langle a, \sigma \rangle$. Then $\langle a, \sigma \rangle \in G_x$. $\square$

Proposition 11.2. *Let $Q \in M_{3,\kappa}$, and let $x \in [\kappa]^\omega$ be P_3-generic over V. Then $x - A \in [\kappa]^{<\omega}$ for some $A \in Q$.*

Proof. Put $X = \{\langle a, \sigma \rangle \in P_3\colon \sigma \in \bigcup_{A \in Q} \Sigma_3^\omega | \ulcorner A \urcorner\}$. It is clear that X is dense in $(P_3, \leqslant)$. Hence $X \cap G_x \neq 0$. $\square$

Corollary 11.3. *Let $B \in \mathfrak{I}$, and let $x \in [\kappa]^\omega$ be P_3-generic over V. Then $x \cap B \in [\kappa]^{<\omega}$.*

Proof. Set $Q = \{\kappa - B\}$, and apply Proposition 11.2. $\square$

$x \in [\kappa]^\omega$ is *rare over V* if for every $g \in V$ with $g : \kappa \to \kappa$, there is an $\alpha \in x$ such that for every $d \in [x - \alpha]^2$, $g(\bigcap d) < \bigcup d$.

Proposition 11.4. *Let $x \in [\kappa]^\omega$ be P_3-generic over V. Then x is rare over V.*

Proof. Given $g \in V$ with $g : \kappa \to \kappa$, let S be the set of all $\sigma \in \Sigma_3^\omega$ such that for every $A \in \langle 0, \sigma \rangle$, $[A]^2 \subseteq \{d \in [\kappa]^2\colon g(\bigcap d) < \bigcup d\}$. Then S is clearly dense in $(\Sigma_3^\omega, \leqslant)$. Hence, $\{\langle a, \sigma \rangle \in P_3\colon \sigma \in S\}$ is dense in $(P_3, \leqslant)$. $\square$

Corollary 11.5. *Let $x \in [\kappa]^\omega$ be P_3-generic over V. Then $\bigcup x = \kappa$.*

Proof. If $x \in [\kappa]^\omega$ is rare over V, then clearly $\bigcup x = \kappa$. $\square$

When combined with Proposition 11.4, the following shows that in case $\kappa = \omega$, forcing with $(P_3, \leqslant)$ adds a dominating real.

Proposition 11.6. *Let $x \in [\omega]^\omega$ be rare over V. Then x is a dominating real over V.*

Proof. Given $f \in V$ with $f\colon \omega \to \omega$, define $g\colon \omega \to \omega$ by letting $g(n) = \bigcup_{q \in n+2} f(q)$. Pick $m \in \omega$ so that for every $d \in [x - e_x(m)]^2$, $g(\bigcap d) < \bigcup d$. Now for every $p \geqslant m$, we have $p + 1 \in e_x(p) + 2$, and therefore $f(p + 1) \leqslant g(e_x(p)) < e_x(p + 1)$. $\square$

The next three propositions will justify our use of the phrase '$P_{\mathfrak{Z}}$-generic set' in connection with the property described at the beginning of this section.

Proposition 11.7. *Let $x \in [\kappa]^\omega$ be $P_{\mathfrak{Z}}$-generic over V. Then G_x is generic for $(P_{\mathfrak{Z}}, \leqslant)$ over V.*

Proof. It clearly suffices to show that any two members of G_x are compatible. Thus $\langle a_i, \sigma_i \rangle \in G_x$ for $i < 2$. Without loss of generality assume that $|a_0| \leqslant |a_1|$. Set $k = |a_1| - |a_0|$, and pick $(g,h) \in P^k_{a_0,\sigma_0}$ with $x \in \langle a_0 \cup ran(g), \sigma_0 | (g,h) \rangle$. Then clearly $a_0 \cup ran(g) = a_i$. Moreover, $x - a_1 \subseteq \sigma_1(0) \cap (\sigma_0 | (g,h))(0)$. Hence, by Corollary 11.3, $\sigma_1(0) \cap (\sigma_0 | (g,h))(0) \in \mathfrak{Z}^+$. By Proposition 4.4, one can find $\sigma_2 \in \Psi(\sigma_0 | (g,h)) \cap \Sigma^\omega_{\mathfrak{Z}} | \sigma_1$. Then $\langle a_1, \sigma_2 \rangle \leqslant \langle a_i, \sigma_i \rangle$ for each $i < 2$. $\square$

Proposition 11.8. *Let G be generic for $(P_{\mathfrak{Z}}, \leqslant)$ over V. Then $\bigcap G = \{x\}$, where $x \in [\kappa]^\omega$ is $P_{\mathfrak{Z}}$-generic over V.*

Proof. Set $x = \{\alpha \in \kappa : \exists \langle a, \sigma \rangle \in G (\alpha \in a)\}$. Notice that $x \in V[G]$. For each $k \in \omega$, put $X_k = \{\langle a, \sigma \rangle \in P_{\mathfrak{Z}} : |a| \geqslant k\}$. X_k is easily seen to be dense in $(P_{\mathfrak{Z}}, \leqslant)$ and, consequently, $X_k \cap G \neq 0$. Therefore, $x \in [\kappa]^\omega$. Clearly, for every $\langle a, \sigma \rangle \in G$, there is an $\alpha \in x$ with $a = x \cap \alpha$. Hence, $\bigcap G \subseteq \{x\}$. Let us now show that $x \in \bigcap G$. Thus, let $\langle a, \sigma \rangle \in G$. By induction on $i < \omega$, we will define $a_i \in [\kappa]^{<\omega}$ and $(g_i, h_i) \in P^{|a_i - a|}_{a,\sigma}$ so that

(0) $a_0 = a$;
(1) $a_i = a \cup ran(g_i)$;
(2) $g_i \subset g_{i+1}$ and $h_i \subset h_{i+1}$;
(3) $\langle a_i, \sigma | (g_i, h_i) \rangle \in G$.

Given i, pick $\langle b, \tau \rangle \in G$ with $|a_i| < |b|$. Then select $\langle b', \tau' \rangle \in G$ so that $\langle b', \tau' \rangle \leqslant \langle b, \tau \rangle$ and $\langle b', \tau' \rangle \leqslant \langle a_i, \sigma | (g_i, h_i) \rangle$. Let $(t, v) \in \bigcup_{k \in \omega} P^k_{a_i, \sigma | (g_i, h_i)}$ be such that $b' = a_i \cup ran(t)$ and $\tau' \in \Phi((\sigma | (g_i, h_i)) | (t, v))$. We set $a_{i+1} = b'$. For every $j < |a_{i+1} - a_i|$, put $g_{i+1}(|a_i - a| + j) = t(j)$ and $h_{i+1}(|a_i - a| + j) = v(j)$. Then $(\sigma | (g_i, h_i)) | (t, v) = \sigma | (g_{i+1}, h_{i+1})$. We have $\langle b', \tau' \rangle \leqslant \langle a_{i+1}, \sigma | (g_{i+1}, h_{i+1}) \rangle$ and, therefore, $\langle a_{i+1}, \sigma | (g_{i+1}, h_{i+1}) \rangle \in G$. Finally, put $g = \bigcup_{i < \omega} g_i$ and $h = \bigcup_{i < \omega} h_i$. Then $(g, h) \in P^\omega_{a,\sigma}$ and $x = a \cup ran(g)$. Hence, $x \in \langle a, \sigma \rangle$. It is clear that x is $P_{\mathfrak{Z}}$-generic over V. $\square$

For G generic for $(P_{\mathfrak{Z}}, \leqslant)$ over V set $x_G = \bigcup \bigcap G$.

Proposition 11.9. (i) *Let G be generic for $(P_{\mathfrak{Z}}, \leqslant)$ over V. Then $G_{x_G} = G$.*
(ii) *Let $x \in [\kappa]^\omega$ be $P_{\mathfrak{Z}}$-generic over V. Then $x_{G_x} = x$.*

Proof. (i): It is clear that $G \subseteq G_{x_G}$. By Proposition 11.8 and Proposition 11.7, G_{x_G} is generic for $(P_{\mathfrak{Z}}, \leqslant)$ over V. Hence, $G = G_{x_G}$.
(ii): Straightforward. $\square$

12. Forcing with $(P_{\mathfrak{I}}, \leqslant)$

The properties of our notion of forcing will now be investigated under the assumption that $\mathfrak{I}$ is almost κ-distributive, which will be used throughout the section (except for Lemma 12.1, Propositions 12.6 and 12.8).

We first show that $(P_{\mathfrak{I}}, \leqslant)$ satisfies what is often referred to as the Prikry property. We will need the following elementary lemma.

Lemma 12.1. *Let $a \in [\kappa]^{<\omega}$ and $C \in \mathfrak{I}^+$, and let ϕ be a sentence of the forcing language of $(P_{\mathfrak{I}}, \leqslant)$ such that for every $\sigma \in \Sigma_{\mathfrak{I}}^{\omega} | \ulcorner C \urcorner$, $\langle a, \sigma \rangle$ does not force ϕ. Then given $A \in \mathfrak{I}^+ \cap P(C)$, there are $\alpha \in \{\beta \in A : a \subseteq \beta\}$ and $B \in \mathfrak{I}^+ \cap P(A - (\alpha + 1))$ such that for every $\sigma' \in \Sigma_{\mathfrak{I}}^{\omega} | \ulcorner B \urcorner$, $\langle a \cup \{\alpha\}, \sigma' \rangle$ does not force ϕ.*

Proof. Assume there is an $A \in \mathfrak{I}^+ \cap P(C)$ with the following property: for every $\alpha \in \{\beta \in A : a \subseteq \beta\}$ and every $B \in \mathfrak{I}^+ \cap P(A - (\alpha + 1))$, there is a $\sigma_{\alpha, B} \in \Sigma_{\mathfrak{I}}^{\omega} | \ulcorner B \urcorner$ such that $\langle a \cup \{\alpha\}, \sigma_{\alpha, B} \rangle$ forces ϕ. Define $\sigma \in \Sigma_{\mathfrak{I}}^{\omega}$ by letting $\sigma(0) = A$ and $\sigma | (\alpha_0, B_0) = \sigma_{\alpha_0, B_0}$. We claim that $\langle a, \sigma \rangle$ forces ϕ. Suppose otherwise. Then there is an $\langle a', \sigma' \rangle \leqslant \langle a, \sigma \rangle$ such that $|a'| > |a|$ and $\langle a', \sigma' \rangle$ forces the negation of ϕ. Pick $(g, h) \in \bigcup_{i < \omega} P_{a,\sigma}^i$ such that $a' = a \cup ran(g)$ and $\sigma' \in \Phi(\sigma | (g, h))$. Then $\langle a', \sigma' \rangle \leqslant \langle a \cup g(0), \sigma_{g(0), h(0)} \rangle$, a contradiction. $\square$

Proposition 12.2. *Assume that $\mathfrak{I}$ is almost κ-distributive. Let $\langle a, \sigma \rangle \in P_{\mathfrak{I}}$, and let ϕ be a sentence of the forcing language of $(P_{\mathfrak{I}}, \leqslant)$. Then there is a $\sigma' \in \Sigma_{\mathfrak{I}}^{\omega} | \sigma$ such that $\langle a, \sigma' \rangle$ forces either ϕ or the negation of ϕ.*

Proof. First assume that there is a $\tau \in \Sigma_{\mathfrak{I}}^{\omega} | \ulcorner \sigma(0) \urcorner$ such that $\langle a, \tau \rangle$ forces ϕ. Then by Proposition 4.4, there is a $\sigma' \in \Sigma_{\mathfrak{I}}^{\omega} | \sigma$ such that $\langle a, \sigma' \rangle \leqslant \langle a, \tau \rangle$. Clearly, $\langle a, \sigma' \rangle$ forces ϕ. Let us now assume otherwise. Define $\Lambda \subseteq \Xi_{\omega}$ by letting $(f, g, h) \in \Lambda$ if and only if there are $i < \omega$ and $\tau \in \Sigma_{\mathfrak{I}}^{\omega} | \ulcorner h(i) \urcorner$ such that $\langle a \cup ran(g | i + 1), \tau \rangle$ forces ϕ. Using Lemma 12.1, it is easy to define a winning strategy for II in $G_{\mathfrak{I}}(a, \sigma(0), \Lambda)$. One readily verifies that given $(f, g, h) \in \Lambda$ and $(f', g, h') \in \Xi_{\omega} - \Lambda$, there is an $i < \omega$ with $h(i) - h'(i) \in \mathfrak{I}^+$. Hence, by Proposition 7.1, there is a $\zeta \in \Sigma_{\mathfrak{I}}^{\omega}$ such that ζ is a winning strategy for I in $G_{\mathfrak{I}}(a, \sigma(0), \Xi_{\omega} - \Lambda)$. It is not difficult to check that $\langle a, \zeta \rangle$ forces the negation of ϕ. By Proposition 4.4, there is a $\sigma' \in \Sigma_{\mathfrak{I}}^{\omega} | \sigma$ such that $\langle a, \sigma' \rangle \leqslant \langle a, \zeta \rangle$. It is clear that $\langle a, \sigma' \rangle$ forces the negation of ϕ. $\square$

Another expected result is that any (infinite) subsequence of a generic sequence is also generic. The proof that follows is a modification of that of Theorem 2.0 in [24].

Proposition 12.3. *Assume that $\mathfrak{I}$ is almost κ-distributive. Let $x \in [\kappa]^{\omega}$ be $P_{\mathfrak{I}}$-generic over V, and let $y \in [x]^{\omega}$. Then y is $P_{\mathfrak{I}}$-generic over V.*

Proof. Let us first work in V. Fix $X \subseteq P_{\mathfrak{I}}$ such that X is open and dense in $(P_{\mathfrak{I}}, \leqslant)$. For each $a \in [\kappa]^{<\omega}$, we define $m_a : \Sigma_{\mathfrak{I}}^{\omega} \to \Sigma_{\mathfrak{I}}^{\omega}$ as follows. Put $k = 2^{|a|} - 1$, and let b_j for $j \leqslant k$ be a one-to-one enumeration of $P(a)$. Given $\sigma \in \Sigma_{\mathfrak{I}}^{\omega}$, use Proposition 7.3 to define $\sigma_j \in \Sigma_{\mathfrak{I}}^{\omega}$ for $j \leqslant k$ so that

(0) $\sigma_0 \leqslant \sigma$;

(1) $\sigma_{r+1} \leqslant \sigma_r$;

(2) for every $(g, h) \in P_{b_j, \sigma_j}^{\omega}$, there is an $i < \omega$ with $\langle b_j \cup ran(g|i), \sigma_j | (g|i, h|i) \rangle \in X$.

Then set $m_a(\sigma) = \sigma_k$. Notice that for every $b \in P(a)$ and every $(g, h) \in P_{b, m_a(\sigma)}^{\omega}$, there is an $i < \omega$ such that $\langle b \cup ran(g|i), (m_a(\sigma)) | (g|i, h|i) \rangle \in X$. We set $Y = \{\langle a, u(m_a(\sigma)) \rangle : a \in [\kappa]^{<\omega}$ and $\sigma \in \Sigma_{\mathfrak{I}}^{\omega}\}$, where $u : \bigcup_{\alpha \in (\omega+1)-1} \Sigma_{\mathfrak{I}}^{\alpha} \to \Sigma_{\mathfrak{I}}^{\omega}$ is as in the proof of Proposition 4.3. Clearly, Y is dense in $(P_{\mathfrak{I}}, \leqslant)$. Now select $a \in [\kappa]^{<\omega}$ and $\sigma \in \Sigma_{\mathfrak{I}}^{\omega}$ so that $\langle a, u(m_a(\sigma)) \rangle \in G_x$. Put $b = a \cap y$. Then $y - b \in \langle 0, m_a(\sigma) \rangle$. Let $(g, h) \in P_{b, m_a(\sigma)}^{\omega}$ be such that $ran(g) = y - b$. In V, let R be the set of all $(t, v) \in \bigcup_{r \in \omega} P_{b, m_a(\sigma)}^{r}$ such that $\langle b \cup ran(t), (m_a(\sigma)) | (t, v) \rangle \notin X$. Given $(t, v), (t', v') \in R$, let $(t, v) \leqslant (t', v')$ just in case $(t', v') = (t|r, v|r)$ for some $r \in \omega$. Clearly, $<$ is a well-founded relation over R. By absoluteness, there is an $i < \omega$ such that $\langle b \cup ran(g|i), (m_a(\sigma)) | (g|i, h|i) \rangle \in X$. Hence by Proposition 11.1, y is $P_{\mathfrak{I}}$-generic over V. □

Our next task consists in showing that forcing with $(P_{\mathfrak{I}}, \leqslant)$ adds many (by which we mean $(2^{\aleph_0})^V$) degrees of constructibility.

Define $\Gamma : [\omega]^{\omega} \times [\kappa]^{\omega} \to [\kappa]^{\omega}$ by letting $\Gamma(d, x) = \{\alpha \in x : |x \cap \alpha| \in d\}$. The following is modeled on Theorem 1.2 in [16].

Proposition 12.4. *Assume that $\mathfrak{I}$ is almost κ-distributive, and let $x \in [\kappa]^{\omega}$ be $P_{\mathfrak{I}}$-generic over V. Then $x \notin V[\Gamma(d, x)]$ for all $d \in V \cap [\omega]^{\omega}$ with $\omega - d \in [\omega]^{\omega}$.*

Proof. Suppose otherwise, and pick $d \in V \cap [\omega]^{\omega}$ such that $\omega - d \in [\omega]^{\omega}$ and $x \in V[\Gamma(d, x)]$. Let $\langle a, \sigma \rangle \in G_x$ and a name z for an element of $V[\Gamma(d, x_G)]$ be such that $\langle a, \sigma \rangle$ forces that $z = x_G$. Let Y be the set of all $\langle a', \sigma' \rangle \in P_{\mathfrak{I}}$ with the following property: either $\langle a', \sigma' \rangle \leqslant \langle a, \sigma \rangle$, or else there is no $\langle a'', \sigma'' \rangle \in P_{\mathfrak{I}}$ such that $\langle a'', \sigma'' \rangle \leqslant \langle a, \sigma \rangle$ and $\langle a'', \sigma'' \rangle \leqslant \langle a', \sigma' \rangle$. Let $u : \bigcup_{\alpha \in (\omega+1)-1} \Sigma_{\mathfrak{I}}^{\alpha} \to \Sigma_{\mathfrak{I}}^{\omega}$ be as in the proof of Proposition 4.3, and let X be the set of all $\langle b, \tau \rangle \in P_{\mathfrak{I}}$ such that for some $\sigma' \in \Sigma_{\mathfrak{I}}^{\omega}$, $\tau = u(\sigma')$ and $\langle b, \sigma' \rangle \in Y$. As X is clearly dense in $(P_{\mathfrak{I}}, \leqslant)$, we have $X \cap G_x \neq 0$. Thus there are $a' \in [\kappa]^{<\omega}$ and $\sigma' \in \Sigma_{\mathfrak{I}}^{\omega}$ such that $\langle a', \sigma' \rangle \in Y$ and $\langle a', u(\sigma') \rangle \in G_x$. Clearly $[x - a']^{\omega} \subseteq \langle 0, \sigma' \rangle$, and $\langle a', \sigma' \rangle$ forces that $z = x_G$. Now select $q \in \omega - d$ with $q \geqslant |a'|$. Put $y_0 = x - \{e_x(q+1)\}$ and $y_1 = x - \{e_x(q)\}$. Clearly, $\Gamma(d, y_0) = \Gamma(d, y_1)$. Moreover, $y_i \in \langle a', \sigma' \rangle$ for $i = 0, 1$. By Proposition 12.3, y_i is $P_{\mathfrak{I}}$-generic over V for $i = 0, 1$. Thus, $\langle a', \sigma' \rangle \in G_{y_i}$ for $i = 0, 1$. But then using Proposition 11.9 (ii), $y_0 = y_1$, a contradiction. □

Corollary 12.5. *Assume that $\mathfrak{I}$ is almost κ-distributive. Let $x \in [\kappa]^{\omega}$ be $P_{\mathfrak{I}}$-generic over V, and let, in $V, Q \in M_{[\omega]^{<\omega}, \omega}$. Then $\Gamma(d', x) \notin V[\Gamma(d, x)]$ for all $d, d' \in Q$ with $d \neq d'$.*

Proof. Assume otherwise, and pick $d, d' \in Q$ such that $d \neq d'$ and $\Gamma(d', x) \in V[\Gamma(d, x)]$. Set $c = \{n \in \omega: e_{d \cup d'}(n) \in d\}$. Clearly, $c \in V \cap [\omega]^\omega$ and $\omega - c \in [\omega]^\omega$. By Proposition 12.3, $\Gamma(d \cup d', x)$ is $P_{\mathfrak{I}}$-generic over V, and so by Proposition 12.4, $\Gamma(d \cup d', x) \notin V[\Gamma(c, \Gamma(d \cup d', x))]$. It is readily verified that $\Gamma(d \cup d', x) = \Gamma(d, x) \cup \Gamma(d', x)$ and $\Gamma(c, \Gamma(d \cup d', x)) = \Gamma(d, x)$. Thus, $\Gamma(d, x) \cup \Gamma(d', x) \notin V[\Gamma(d, x)]$, which yields the desired contradiction. $\square$

The forthcoming Proposition 12.7 shows that a well-known property of Prikry forcing is likewise enjoyed by the notion of forcing $(P_{\mathfrak{I}}, \leqslant)$. We will not explicitly require κ to be uncountable, although the result is trivial, and thus uninteresting, in case $cov(\mathfrak{I}) = \omega$. We first introduce some notation.

Assuming $add(\mathfrak{I}) = cov(\mathfrak{I})$, let $R_\beta \in \mathfrak{I}$ for $\beta < add(\mathfrak{I})$ be fixed such that $\bigcup_{\beta < add(\mathfrak{I})} R_\beta = \kappa$, $R_\beta \cap R_\gamma = 0$ whenever $\beta \neq \gamma$ and in case $add(\mathfrak{I}) = \kappa$, $R_\beta = \{\beta\}$ for all β. Then given an $x \in [\kappa]^\omega$ such that x is $P_{\mathfrak{I}}$-generic over V, set $t_x = \{\beta \in add(\mathfrak{I}): x \cap R_\beta \neq 0\}$.

We observe that $t_x = x$ in case $add(\mathfrak{I}) = \kappa$.

Proposition 12.6. *Assume that $add(\mathfrak{I}) = cov(\mathfrak{I})$, and let $x \in [\kappa]^\omega$ be $P_{\mathfrak{I}}$-generic over V. Then $\bigcup t_x = add(\mathfrak{I})$.*

Proof. Given $\beta \in add(\mathfrak{I})$, $\{\langle a, \sigma\rangle \in P_{\mathfrak{I}}: \sigma(0) \cap \bigcup_{\gamma \leqslant \beta} R_\gamma = 0\}$ is clearly dense in $(P_{\mathfrak{I}}, \leqslant)$. $\square$

The following is analogous to Corollary 2.15 of [28].

Proposition 12.7. *Assume that $add(\mathfrak{I}) = cov(\mathfrak{I})$ and $\mathfrak{I}$ is almost κ-distributive. Let $x \in [\kappa]^\omega$ be $P_{\mathfrak{I}}$-generic over V, and let, in $V[x]$, $f: add(\mathfrak{I}) \to On$. Then there are $f_n \in V$ for $n \in \omega$ with $f = \bigcup_{n \in \omega} f_n$.*

Proof. Let $\langle a, \sigma\rangle \in G_x$ be such that $\langle a, \sigma\rangle$ forces that $f: add(\mathfrak{I}) \to On$. Given $b \in [\kappa]^{<\omega}$, we define $H_b: \Sigma_{\mathfrak{I}}^\omega \to \Sigma_{\mathfrak{I}}^\omega$ as follows. Let $\tau \in \Sigma_{\mathfrak{I}}^\omega$. First assume that $b \neq 0$ and $\langle b, \tau\rangle \leqslant \langle a, \sigma\rangle$. Set $\delta = \bigcup \{\beta \in add(\mathfrak{I}): b \cap R_\beta \neq 0\}$. Given $\gamma \in \delta$, let T_γ be the set of all $\rho \in \Sigma_{\mathfrak{I}}^\omega | \tau$ such that for some $\zeta \in On$, $\langle b, \rho\rangle$ forces that $f(\gamma) = \zeta$. Set $K_\gamma = \{Q \in D_{\mathfrak{I}, \tau(0)}: Q \subseteq \{\rho(0): \rho \in T_\gamma\}\}$, and let Q_γ be a maximal element of $(K_\gamma, \subseteq)$. Then select $Z_\gamma \in M_{\mathfrak{I}, \tau(0)}$ with $Q_\gamma \subseteq Z_\gamma$. By Proposition 3.3, there is a $k \in \prod_{\gamma < \delta} Z_\gamma$ with $\bigcap_{\gamma \in \delta} k(\gamma) \in \mathfrak{I}^+$. Define $\sigma_\gamma \in \Sigma_{\mathfrak{I}}^\omega$ for $\gamma \in \delta$ so that

(0) in case $k(\gamma) \in Q_\gamma$, $\sigma_\gamma \in T_\gamma$ and $\sigma_\gamma(0) = k(\gamma)$;

(1) otherwise $\sigma_\gamma = \ulcorner k(\gamma) \urcorner$.

Using the proof of Proposition 4.5, find $\tau' \in \Sigma_{\mathfrak{I}}^\omega | \tau \cap \bigcap_{\gamma \in \delta} \Psi(\sigma_\gamma)$, and set $H_b(\tau) = \tau'$. Now assume that $b = 0$, or that it is not the case that $\langle b, \tau\rangle \leqslant \langle a, \sigma\rangle$. Then we put $H_b(\tau) = \tau$. Given $n \in \omega$, let Y_n be the set of all $\langle b, \tau\rangle \in P_{\mathfrak{I}}$ such that $|b| > n$ and either $\langle b, \tau\rangle \leqslant \langle a, \sigma\rangle$, or else $\langle b, \tau\rangle$ and $\langle a, \sigma\rangle$ are incompatible in $(P_{\mathfrak{I}}, \leqslant)$. We put $X_n = \{\langle b, H_b(\tau)\rangle: \langle b, \tau\rangle \in Y_n\}$. X_n is clearly dense in $(P_{\mathfrak{I}}, \leqslant)$. Select $\langle b_n, \tau_n\rangle \in Y_n$

such that $\langle b_n, H_{b_n}(\tau_n)\rangle \in G_x$. It is clear that $\langle b_n, H_{b_n}(\tau_n)\rangle \leqslant \langle b_n, \tau_n\rangle \leqslant \langle a, \sigma\rangle$. Set $\xi_n = \bigcup \{\beta \in add(\mathfrak{I})\colon b_n \cap R_\beta \neq 0\}$, and let f_n be the set of all $(\gamma, \zeta) \in \xi_n \times On$ such that $\langle b_n, H_{b_n}(\tau_n)\rangle$ forces that $f(\gamma) = \zeta$. Clearly, $f_n \in V$, and $f_n \subset f$. Now fix $\gamma \in add(\mathfrak{I})$. Let $\langle a', \sigma'\rangle \in G_x$ and $\zeta \in On$ be such that $\langle a', \sigma'\rangle$ forces that $f(\gamma) = \zeta$. By Proposition 12.6, $\bigcup_{n \in \omega} \xi_n = add(\mathfrak{I})$. Choose $n \in \omega$ so that $\gamma \in \xi_n$ and $a' \subseteq b_n$. By the proof of Proposition 11.7, there is a $\sigma'' \in \Sigma_{\mathfrak{I}}^{\omega} | H_{b_n}(\tau_n)$ such that $\langle b_n, \sigma''\rangle \leqslant \langle a', \sigma'\rangle$. Clearly, $\langle b_n, \sigma''\rangle$ forces that $f(\gamma) = \zeta$. It is readily checked that $\langle b_n, H_{b_n}(\tau_n)\rangle$ forces that $f(\gamma) = \zeta$. Thus $f = \bigcup_{n \in \omega} f_n$. □

We conclude the section by a study of the preservation of cardinals under forcing with $(P_{\mathfrak{I}}, \leqslant)$. We start with this easy observation.

Proposition 12.8. *$(P_{\mathfrak{I}}, \leqslant)$ satisfies the $(\kappa^+ \cdot sat(\mathfrak{I}))$-chain condition.*

Proof. Use Proposition 4.4. □

Proposition 12.9. *Assume that $\mathfrak{I}$ is almost κ-distributive. Then forcing with $(P_{\mathfrak{I}}, \leqslant)$ adds no new bounded subset of $add(\mathfrak{I})$.*

Proof. Fix an ordinal δ with $0 < \delta < add(\mathfrak{I})$, and let $\langle a, \sigma\rangle \in P_{\mathfrak{I}}$ and F be such that $\langle a, \sigma\rangle$ forces that $F: \delta \to 2$. For each $\gamma < \delta$, let S_γ be the set of all $\tau \in \Sigma_{\mathfrak{I}}^{\omega} | \sigma$ such that for some $i < 2$, $\langle a, \tau\rangle$ forces that $F(\gamma) = i$. By Propositions 12.2, 3.3 and 4.5, one can find $\sigma' \in \Sigma_{\mathfrak{I}}^{\omega} | \sigma \cap (\bigcap_{\gamma \in \delta} \bigcup_{\tau \in S_\gamma} \Psi(\tau))$. In V, define $f: \delta \to 2$ by letting $f(\gamma) = i$ if and only if $\langle a, \sigma'\rangle$ forces that $F(\gamma) = i$. It is clear that $\langle a, \sigma'\rangle$ forces that $F = f$. □

The main (expected) preservation result concerns κ^+. It will be derived from Proposition 12.11, which is interesting in its own right. We start with this lemma.

Lemma 12.10. *Assume that $\mathfrak{I}$ is κ-distributive and $add(\mathfrak{I}) = \kappa$. Let $\langle a, \sigma\rangle \in P_{\mathfrak{I}}$ and F be such that $\langle a, \sigma\rangle$ forces that $F: \kappa \to V$, and let $\alpha \in \kappa$. Then there exists $\sigma' \in \Sigma_{\mathfrak{I}}^{\omega} | \sigma$ with the following property: for every $(g, h) \in P_{a,\sigma'}^{\omega}$ and every $\gamma \in \alpha$, there are $i \in \omega$ and $z \in V$ such that $\langle a \cup ran(g | i), \sigma' | (g | i, h | i)\rangle$ forces that $F(\gamma) = z$.*

Proof. Given $\gamma \in \alpha$, let $Y_\gamma \subseteq P_{\mathfrak{I}}$ be defined by letting $\langle b, \tau\rangle \in Y_\gamma$ if and only if either $\langle b, \tau\rangle$ and $\langle a, \sigma\rangle$ are incompatible in $(P_{\mathfrak{I}}, \leqslant)$, or else $\langle b, \tau\rangle \leqslant \langle a, \sigma\rangle$ and there is a $z \in V$ such that $\langle b, \tau\rangle$ forces that $F(\gamma) = z$. Also let $S_\gamma \subseteq \Sigma_{\mathfrak{I}}^{\omega}$ be defined by letting $\tau \in S_\gamma$ if and only if $\tau \leqslant \sigma$ and for every $(g, h) \in P_{a,\tau}^{\omega}$, there are $i \in \omega$ and $z \in V$ such that $\langle a \cup ran(g | i), \tau | (g | i, h | i)\rangle$ forces that $F(\gamma) = z$. Each Y_γ is readily verified to be open and dense in $(P_{\mathfrak{I}}, \leqslant)$. Hence, by Proposition 7.3 and Proposition 4.5, one can find $\sigma' \in \Sigma_{\mathfrak{I}}^{\omega} | \sigma \cap (\bigcap_{\gamma \in \alpha} \bigcup_{\tau \in S_\gamma} \Psi(\tau))$. It is not difficult to check that σ' is as desired. □

Proposition 12.11. *Assume that $\mathfrak{I}$ is κ-distributive and $add(\mathfrak{I}) = \kappa$. Let $\langle a, \sigma \rangle \in P_{\mathfrak{I}}$ and F be such that $\langle a, \sigma \rangle$ forces that $F : \kappa \to V$. Then there exist $\sigma' \in \Sigma^{\omega}_{\mathfrak{I}} | \sigma$ and $A \in V$ such that $|A| \leqslant \kappa$ and $\langle a, \sigma' \rangle$ forces that $ran(F) \subseteq A$.*

Proof. Let us first define $\tau \in \Sigma^{\omega}_{\mathfrak{I}} | \sigma$ as follows. Put $\tau(0) = \{\beta \in \sigma(0): a \subseteq \beta\}$. Now let II play (α_i, B_i) for $i < \omega$. We use Lemma 12.10 to define $\rho_i \in \Sigma^{\omega}_{\mathfrak{I}}$ for $i < \omega$ so that

(0) $\rho_0 \leqslant \sigma | (\alpha_0, B_0)$;

(1) $\rho_{i+1} \leqslant \rho_i | (\alpha_{i+1}, B_{i+1})$;

(2) for every $(g, h) \in P^{\omega}_{a \cup \{\alpha_k : k \leqslant i\}, \rho_i}$ and every $\gamma \in \alpha_i$, there are $j \in \omega$ and $z \in V$ such that $\langle a \cup \{\alpha_k: k \leqslant i\} \cup ran(g | j), \rho_i | (g | j, h | j) \rangle$ forces that $F(\gamma) = z$.

We put $\tau((\alpha_0, B_0), \ldots, (\alpha_i, B_i)) = \rho_i(0)$. Clearly, given $(g, h) \in P^{\omega}_{a, \tau}$, $i \in \omega$ and $\gamma \in g(i)$, there are $j \in \omega$ and $z \in V$ such that $\langle a \cup ran(g | j), \tau | (g | j, h | j) \rangle$ forces that $F(\gamma) = z$. Given $(g, h) \in \bigcup_{j \in \omega} P^{j}_{a, \tau}$, let $E_{g,h}$ be the set of all $\gamma \in \kappa$ for which one can find $z \in V$ such that $\langle a \cup ran(g), \tau | (g, h) \rangle$ forces that $F(\gamma) = z$. Further let $f_{g,h} : E_{g,h} \to V$ be such that for every $\gamma \in E_{g,h}$, $\langle a \cup ran(g), \tau | (g, h) \rangle$ forces that $F(\gamma) = f_{g,h}(\gamma)$. For each $i \in \omega$, let T_i be the set of all $g \in \kappa^i$ such that the following holds:

$$g(0) \in (\varepsilon_1(\tau))(0), \quad \text{and} \quad g(j+1) \in (\varepsilon_1(\tau))(g(0), \ldots, g(j)).$$

Let $w : \bigcup_{k \in \omega} T_{k+1} \to \mathfrak{I}^+$ be such that for every $k \in \omega$ and every $g \in T_{k+1}$,

$$(\varepsilon_1(\tau))(g(0), \ldots, g(k)) \subseteq \tau((g(0), w(g | 1)), \ldots, (g(k), w(g))).$$

Then define $H : \bigcup_{k \in \omega} T_{k+1} \to \bigcup_{k \in \omega} (\mathfrak{I}^+)^{k+1}$ so that given $k \in \omega$ and $g \in T_{k+1}$, we have $H(g) \in (\mathfrak{I}^+)^{k+1}$ and for every $j \in k+1$, $(H(g))(j) = w(g | (j+1))$. For each $\gamma \in \kappa$, define $Z_\gamma \subseteq V$ by letting $z \in Z_\gamma$ if and only if there is a $g \in \bigcup_{k \in \omega} T_{k+1}$ such that $\gamma \in E_{g, H(g)}$ and $f_{g, H(g)}(\gamma) = z$. Notice that $|Z_\gamma| \leqslant \kappa$. It is readily checked that given $(g, h) \in P^{\omega}_{a, \varepsilon_0(\varepsilon_1(\tau))}$, $i \in \omega$ and $\gamma \in g(i)$, there is a $k \in \omega$ such that $\langle a \cup ran(g | (k+1)), (\varepsilon_0(\varepsilon_1(\tau))) | (g | (k+1), h | (k+1)) \rangle$ forces that $F(\gamma) = f_{g | (k+1), H(g | (k+1))}(\gamma)$. Now let $\gamma \in \kappa$, and let $\langle a', \sigma' \rangle \in P_{\mathfrak{I}}$ be such that $\langle a', \sigma' \rangle \leqslant \langle a, \varepsilon_0(\varepsilon_1(\tau)) \rangle$. Pick $(g, h) \in P^{\omega}_{a', \sigma'}$ so that $g(0) > \gamma$. It is easily seen that there is a $j < \omega$ such that $\langle a' \cup ran(g | j), \sigma' | (g | j, h | j) \rangle$ forces that $F(\gamma) \in Z_\gamma$. Thus there is an $\langle a'', \sigma'' \rangle \leqslant \langle a', \sigma' \rangle$ such that $\langle a'', \sigma'' \rangle$ forces that $F(\gamma) \in Z_\gamma$. Hence for every $\gamma \in \kappa$, $\langle a, \varepsilon_0(\varepsilon_1(\tau)) \rangle$ forces that $F(\gamma) \in Z_\gamma$. $\square$

The following is now straightforward.

Corollary 12.12. *Assume that $\mathfrak{I}$ is κ-distributive and $add(\mathfrak{I}) = \kappa$. Then forcing with $(P_{\mathfrak{I}}, \leqslant)$ preserves κ^+.*

It follows from the results above that no cardinals are collapsed when forcing with $(P_{\mathfrak{I}}, \leqslant)$ provided that $\mathfrak{I}$ is κ-distributive and satisfies $add(\mathfrak{I}) = \kappa$ and $sat(\mathfrak{I}) \leqslant \kappa^{++}$.

13. Ramsey subsets of κ

$x \in [\kappa]^\omega$ is *Ramsey over* V if for every $F \in V$ with $F : [\kappa]^2 \to 2$, there exists $\alpha \in x$ such that F is constant on $[x - \alpha]^2$.

Let us first make this easy observation.

Proposition 13.1. *Let $x \in [\kappa]^\omega$ be Ramsey over V and such that $\bigcup x = \kappa$. Then x is rare over V.*

Proof. Given $g \in V$ with $g : \kappa \to \kappa$, define $F : [\kappa]^2 \to 2$ by letting $F(d) = 0$ if and only if $g(\bigcap d) < \bigcup d$. □

Lemma 13.2. *Assume that $\mathfrak{I}$ is almost κ-distributive. Let $S \subseteq \Sigma_{\mathfrak{I}}^\omega$ be dense in $(\Sigma_{\mathfrak{I}}^\omega, \leqslant)$, and let $x \in [\kappa]^\omega$ be $P_{\mathfrak{I}}$-generic over V. Then there are $\sigma \in S$ and $\alpha \in x$ with $[x - \alpha]^\omega \subseteq \langle 0, \sigma \rangle$.*

Proof. Let $u : \bigcup_{\alpha \in (\omega + 1) - 1} \Sigma_{\mathfrak{I}}^\alpha \to \Sigma_{\mathfrak{I}}^\omega$ be as in the proof of Proposition 4.3. Put $X = \{\langle a, u(\sigma)\rangle : a \in [\kappa]^{<\omega}$ *and* $\sigma \in S\}$. Then clearly X is dense in $(P_{\mathfrak{I}}, \leqslant)$. Let $a \in [\kappa]^{<\omega}$ and $\sigma \in S$ be such that $\langle a, u(\sigma)\rangle \in G_x$. We have $[x - a]^\omega \subseteq \langle 0, \sigma \rangle$. □

Proposition 13.3. *Assume that $\mathfrak{I}$ is almost κ-distributive, and let $x \in [\kappa]^\omega$ be $P_{\mathfrak{I}}$-generic over V. Then x is Ramsey over V.*

Proof. Fix $W \in V$ with $W \subseteq [\kappa]^2$. Let S be the set of all $\sigma \in \Sigma_{\mathfrak{I}}^\omega$ such that either $\langle 0, \sigma \rangle \subseteq \{A \in [\kappa]^\omega : [A]^2 \subseteq W\}$, or else $\langle 0, \sigma \rangle \subseteq \{A \in [\kappa]^\omega : [A]^2 \cap W = 0\}$. By Propositions 7.1, 4.3 and 4.4, S is dense in $(\Sigma_{\mathfrak{I}}^\omega, \leqslant)$. Hence, by Lemma 13.2, there are $\alpha \in x$ and $\sigma \in S$ such that $x - \alpha \in \langle 0, \sigma \rangle$. Then either $[x - \alpha]^2 \subseteq W$, or else $[x - \alpha]^2 \cap W = 0$. □

Let us also mention the following related result, the proof of which is modeled on that of Theorem 1.15 in [13].

Proposition 13.4. *Assume that $\kappa = \omega$ and $\mathfrak{I}$ is $\aleph_0$-distributive. Let ϕ be a Σ_2^1 formula with parameters in V, and let $x \in [\omega]^\omega$ be $P_{\mathfrak{I}}$-generic over V. Then in $V[x]$, there is an $\alpha \in x$ such that either $[x - \alpha]^\omega \subseteq \{y : \phi(y)\}$, or else $[x - \alpha]^\omega \cap \{y : \phi(y)\} = 0$.*

Proof. By Proposition 12.2 and Lemma 13.2, there are $\sigma \in \Sigma_{\mathfrak{I}}^\omega$, $\alpha \in x$ and ψ such that the following holds: (i) ψ is either ϕ or the negation of ϕ; (ii) $\langle 0, \sigma \rangle$ forces $\psi(x_G)$; (iii) $[x - \alpha]^\omega \subseteq \langle 0, \sigma \rangle$. Given $y \in [x - \alpha]^\omega \cap V[x]$, y is $P_{\mathfrak{I}}$-generic over V by Proposition 12.3. Clearly $\langle 0, \sigma \rangle \in G_y$, and so by Proposition 11.9 (ii), $V[y]$ satisfies $\psi(y)$. We have $V[y] \subseteq V[x]$, and therefore by absoluteness, $V[x]$ satisfies $\psi(y)$. □

We devote the remainder of this section to two related problems. One of them is to find a nice combinatorial characterization of $P_{\mathfrak{I}}$-generic sets. The other one is to show that forcing with $(P_{\mathfrak{I}}, \leqslant)$ is equivalent to forcing with some partially ordered set with a nicer definition. We will see that the answers become simpler as we require $\mathfrak{I}$ to satisfy stronger properties.

Proposition 13.5. *Assume that $\mathfrak{I}$ is κ-distributive and $add(\mathfrak{I}) = \kappa$. Then given $x \in [\kappa]^\omega$, the following are equivalent:*

(i) *x is $P_{\mathfrak{I}}$-generic over V.*

(ii) *x is Ramsey over V and for every $Q \in V$ with $Q \in M_{\mathfrak{I},\kappa}$, $x \in \bigcup_{a \in [\kappa]^{<\omega}} \bigcup_{A \in Q} \langle a, A \rangle$.*

(iii) *$x \in \bigcup_{a \in [\kappa]^{<\omega}} \bigcup_{\sigma \in S} \langle a, \sigma \rangle$ whenever $S \subseteq \Sigma_{\mathfrak{I}}^\omega$ is dense in $(\Sigma_{\mathfrak{I}}^\omega, \leqslant)$.*

Proof. (i) $\to$ (ii): By Proposition 13.3 and Proposition 11.2.

(ii) $\to$ (iii): Assume (ii), and let $S \subseteq \Sigma_{\mathfrak{I}}^\omega$ be dense in $(\Sigma_{\mathfrak{I}}^\omega, \leqslant)$. Set $Y = \{R \in D_{\mathfrak{I},\kappa}: R \subseteq \{\sigma(0): \sigma \in S\}\}$. Let R be a maximal element of $(Y, \subseteq)$. It is readily verified that $R \in M_{\mathfrak{I},\kappa}$. Select $T \subseteq S$ such that $\{\sigma(0): \sigma \in T\} = R$, and that $\sigma(0) \cap \sigma'(0) \in \mathfrak{I}$ for all $\sigma, \sigma' \in T$ with $\sigma \neq \sigma'$. For each $\sigma \in T$, let $Q_\sigma \in M_{\mathfrak{I},\sigma(0)}$ be such that for all $a \in [\sigma(0)]^{<\omega} - \{0\}$, $Q_\sigma \leqslant Q_a^\sigma$. Clearly, $\bigcup_{\sigma \in T} Q_\sigma \in M_{\mathfrak{I},\kappa}$. Now let $\sigma \in T$ and $A \in Q_\sigma$ be such that $x - A \in [\kappa]^{<\omega}$. Furthermore, let $f \in \prod_{a \in [\sigma(0)]^{<\omega} - \{0\}} Q_a^\sigma$ and $g: [\sigma(0)]^{<\omega} - \{0\} \to \mathfrak{I}$ be such that $A - f(a) = g(a)$. Define $F: [\kappa]^2 \to 2$ by letting $F(d) = 0$ if and only if for some $a \in [((\bigcap d) + 1) \cap \sigma(0)]^{<\omega} - \{0\}$, $\bigcup d \in g(a)$. Let $\alpha \in x$ and $i < 2$ be such that $x - \alpha \subseteq A$ and F is identically i on $[x - \alpha]^2$. Then $\bigcup_{a \in [(\alpha+1) \cap \sigma(0)]^{<\omega} - \{0\}} g(a) \in \mathfrak{I}$. Hence by Corollary 11.3, $i = 1$. It easily follows that $x \in \langle x \cap \alpha, \sigma \rangle$.

(iii) $\to$ (i): Assume (iii), and let $X \subseteq P_{\mathfrak{I}}$ be dense and open in $(P_{\mathfrak{I}}, \leqslant)$. Let S be the set of all $\sigma \in \Sigma_{\mathfrak{I}}^\omega$ such that the following holds: for every $a \in [\kappa]^{<\omega}$ and every $(g, h) \in P_{a,\sigma}^\omega$, there is an $i < \omega$ with $\langle a \cup ran(g \mid i), \sigma \mid (g \mid i, h \mid i) \rangle \in X$. By Proposition 7.5, S is dense in $(\Sigma_{\mathfrak{I}}^\omega, \leqslant)$. Let $\sigma \in S$ and $a \in [\kappa]^{<\omega}$ be such that $x \in \langle a, \sigma \rangle$. In V, let R be the set of all $(t, v) \in \bigcup_{r \in \omega} P_{a,\sigma}^r$ such that $\langle a \cup ran(t), \sigma \mid (t, v) \rangle \notin X$. Given $(t, v), (t', v') \in R$, let $(t, v) \leqslant (t', v')$ just in case $(t', v') = (t \mid r, v \mid r)$ for some $r \in \omega$. Then $<$ is a well-founded relation over R. By absoluteness, there is a $(t, v) \in \bigcup_{r \in \omega} P_{a,\sigma}^r$ such that $x \in \langle a \cup ran(t), \sigma \mid (t, v) \rangle \in X$. Hence by Proposition 11.1, x is $P_{\mathfrak{I}}$-generic over V. $\square$

Proposition 13.6. *Assume either that $\mathfrak{I}$ is κ-distributive, or else that $\mathfrak{I}$ is almost κ-distributive and $add(\mathfrak{I}) > \aleph_0$. Then $(P_{\mathfrak{I}}, \leqslant)$ and $(P_{\mathfrak{I}}^*, \subseteq)$ yield the same generic extensions.*

Proof. We recall that ε_0 and ε_1 were defined in Section 9. Now define $i: P_{\mathfrak{I}}^* \to P_{\mathfrak{I}}$ by letting $i(\langle a, \sigma \rangle^*) = \langle a, \varepsilon_0(\sigma) \rangle$. Given $\langle a, \sigma \rangle \in P_{\mathfrak{I}}$, we have $\langle a, \varepsilon_0(\varepsilon_1(\sigma)) \rangle \leqslant \langle a, \sigma \rangle$. Thus $ran(i)$ is dense in $(P_{\mathfrak{I}}, \leqslant)$. It is not difficult to verify that if $\langle a', \sigma' \rangle \leqslant \langle a, \varepsilon_0(\sigma) \rangle$, then $\langle a', \varepsilon_1(\sigma') \rangle^* \subseteq \langle a, \sigma \rangle^*$. Hence, $i(\langle a, \sigma \rangle^*)$ and $i(\langle a', \sigma' \rangle^*)$ are incompatible in $(P_{\mathfrak{I}}, \leqslant)$ whenever $\langle a, \sigma \rangle^*$ and $\langle a', \sigma' \rangle^*$ are incompatible in $(P_{\mathfrak{I}}^*, \subseteq)$. Moreover,

$\langle a', \sigma' \rangle^* \subseteq \langle a, \sigma \rangle^*$ easily implies that $i(\langle a', \sigma' \rangle^*) \leqslant i(\langle a, \sigma \rangle^*)$. Thus, i is a dense embedding in the sense of Definition 7.7 of Chapter VII of [19]. Now apply Theorem 7.11 in Chapter VII of [19] to obtain the desired result. $\square$

Proposition 13.7. *Assume that $\mathfrak{I}$ is a κ-distributive weak P-point. Then given $x \in [\kappa]^\omega$, the following are equivalent:*
(i) *x is $P_{\mathfrak{I}}$-generic over V.*
(ii) *x is rare over V and for every $Q \in V$ with $Q \in M_{\mathfrak{I},\kappa}$, $x \in \bigcup_{a \in [\kappa]^{<\omega}} \bigcup_{A \in Q} \langle a, A \rangle$.*

Proof. (i) $\to$ (ii): By Propositions 11.2 and 11.4.

(ii) $\to$ (i): Assume (ii), and fix $F \in V$ with $F: [\kappa]^2 \to 2$. For every $\beta \in \kappa$ and every $i < 2$, set $E^i_\beta = \{\gamma \in \kappa - (\beta + 1): F(\beta, \gamma) = i\}$. For each $\beta \in \kappa$, put $Q_\beta = \{E^0_\beta, E^1_\beta\} \cap \mathfrak{I}^+$. Select $Q \in M_{\mathfrak{I},\kappa}$ so that $Q \leqslant Q_\beta$ for all $\beta \in \kappa$. Let $f: Q \to 2^\kappa$ be such that $A - E^{(f(A))(\beta)}_\beta \in \mathfrak{I}$. Given $A \in Q$, let Y be the set of all $R \subseteq D_{\mathfrak{I},A}$ such that for every $\beta \in \kappa$ and every $B \in R$, $B - E^{(f(A))(\beta)}_\beta \in [\kappa]^{<\kappa}$. Let R_A be a maximal element of $(Y, \subseteq)$. By Proposition 1.7, $R_A \in M_{\mathfrak{I},A}$. For each $i < 2$, set $R^i_A = \{B \cap (f(A))^{-1}(\{i\}): B \in R_A\} \cap \mathfrak{I}^+$. It is clear that $\bigcup_{A \in Q} \bigcup_{i<2} R^i_A \in M_{\mathfrak{I},\kappa}$. Now select $A \in Q$, $i < 2$, $B \in R^i_A$ and $\alpha \in x$ such that $x - \alpha \subseteq B$. Define $g: \kappa \to \kappa$ by letting $g(\beta) = \bigcup (B - E^{(f(A))(\beta)}_\beta)$. Let $\delta \in x$ be such that for every $d \in [x - \delta]^2$, $g(\bigcap d) < \bigcup d$. Then F is identically i on $[x - (\alpha \cup \delta)]^2$. Thus, x is Ramsey over V. Hence by Proposition 13.5, x is $P_{\mathfrak{I}}$-generic over V. $\square$

We observe that by Lemma 10.1, if $\mathfrak{I}$ is a weak P-point, then $(P^*_{\mathfrak{I}}, \subseteq)$ and $(P^*_{[\kappa]^{<\kappa},\mathfrak{I}}, \subseteq)$ yield the same generic extensions.

Proposition 13.8. *Assume that $\mathfrak{I}$ is κ-distributive and weakly selective. Then given $x \in [\kappa]^\omega$, the following are equivalent:*
(i) *x is $P_{\mathfrak{I}}$-generic over V.*
(ii) *$x \in \bigcup_{a \in [\kappa]^{<\omega}} \bigcup_{A \in Q} \langle a, A \rangle$ for every $Q \in V$ with $Q \in M_{\mathfrak{I},\kappa}$.*

Proof. (i) $\to$ (ii): By Proposition 11.2.

(ii) $\to$ (i): Assume (ii), and fix $g \in V$ with $g: \kappa \to \kappa$. By Lemma 1.8 there is a $Q \in M_{\mathfrak{I},\kappa}$ such that for every $A \in Q$ and every $\beta \in A$, $\beta \notin \bigcup_{\gamma \in A \cap \beta} (g(\gamma) + 1)$. Let $A \in Q$ and $\alpha \in x$ be such that $x - \alpha \subseteq A$. Clearly, $g(\bigcap d) < \bigcup d$ for every $d \in [x - \alpha]^2$. Hence, by Proposition 13.7, x is $P_{\mathfrak{I}}$-generic over V. $\square$

$(Q_{\mathfrak{I}}, \subseteq)$ and $(\{\langle a, \lfloor C \rfloor \rangle^*: a \in [\kappa]^{<\omega}$ *and* $C \in \mathfrak{I}^+\}, \subseteq)$ are clearly isomorphic. Hence if $\mathfrak{I}$ is a weak Q-point, then by Lemma 10.19, $(P^*_{[\kappa]^{<\kappa},\mathfrak{I}}, \subseteq)$ and $(Q_{\mathfrak{I}}, \subseteq)$ yield the same generic extensions.

Let us recall the following. Assume that $\kappa > \omega$, $\mathfrak{I}$ is prime and $add(\mathfrak{I}) = \kappa$. Then by the results of the previous section forcing with $(P_{\mathfrak{I}}, \leqslant)$ (i) preserves all cardinals, (ii) changes the cofinality of κ to ω and (iii) does not add any new bounded subset of κ. By [8] the notion of forcing $(Q_{\mathfrak{I}}, \subseteq)$ satisfies (i)–(iii) if and only if $\mathfrak{I}$ is weakly selective.

This allows one to argue that $(P_{\mathfrak{I}}, \leqslant)$ is the correct analog to Prikry forcing in case $\mathfrak{I}$ is not weakly selective.

14. Forcing with $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$

The main goal of this section is to obtain the decomposition of the notion of forcing $(P_{\mathfrak{I}}, \leqslant)$ as a two step iteration (see Propositions 14.3–14.5). The result will be secured under the assumption that $\mathfrak{I}$ is κ-distributive and $add(\mathfrak{I}) = \kappa$. There are however on the way some partial results that require less and will be stated in full generality. We start by making a few observations concerning forcing with $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$.

Given $H \subseteq \mathfrak{I}^+/\mathfrak{I}$ such that H is generic for $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$ over V, we set

$$\mathfrak{I}_H = \{A \in P(\kappa) \cap V\colon [\kappa - A]_{\mathfrak{I}} \in H\}.$$

The following is well-known.

Proposition 14.1. *Let $H \subseteq \mathfrak{I}^+/\mathfrak{I}$ be such that H is generic for $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$ over V. Then the following assertions hold*:

(0) $\mathfrak{I} \subseteq \mathfrak{I}_H$ *and* $\mathfrak{I}_H \cap \mathfrak{I}^+ = 0$.

(1) $P(A) \cap V \subseteq \mathfrak{I}_H$ *for all* $A \in \mathfrak{I}_H$.

(2) $A \cup B \in \mathfrak{I}_H$ *whenever* $A, B \in \mathfrak{I}_H$.

(3) *Given* $A \in P(\kappa) \cap V$, *either* $A \in \mathfrak{I}_H$, *or else* $\kappa - A \in \mathfrak{I}_H$.

(4) $add(\mathfrak{I}_H) \leqslant cov(\mathfrak{I})$.

(5) *In V, let μ be an infinite cardinal $< add(\mathfrak{I})$, and let $A_\alpha \subseteq \kappa$ for $\alpha < \mu$. If each $A_\alpha \in \mathfrak{I}_H$, then $\bigcup_{\alpha < \mu} A_\alpha \in \mathfrak{I}_H$.*

(6) *If $\mathfrak{I}$ is κ-distributive and a weak P-point* (*respectively weak Q-point*), *then $\mathfrak{I}_H$ is a weak P-point* (*resp. weak Q-point*).

The following is well-known in case $\mathfrak{I} = [\omega]^{<\omega}$ (see [1]).

Proposition 14.2. *Assume that $\mathfrak{I}$ is nowhere $\boldsymbol{h}_{\mathfrak{I}}$-distributive, $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$ has a dense κ-closed subset and $2^{<\kappa} < \boldsymbol{h}_{\mathfrak{I}}$. Let $H \subseteq \mathfrak{I}^+/\mathfrak{I}$ be such that H is generic for $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$ over V. Then $(2^\kappa)^{V[H]} = (\boldsymbol{h}_{\mathfrak{I}})^V$.*

Proof. Use Propositions 2.14 and 2.15 and follow the proof of Proposition 13.3 in [23]. $\square$

Let $x \in [\kappa]^\omega$ be $P_{\mathfrak{I}}$-generic over V. We set

$$\mathfrak{I}_x = \{A \in V \cap P(\kappa)\colon x \cap A \in [\kappa]^{<\omega}\} \quad \text{and} \quad H_x = \{[\kappa - A]_{\mathfrak{I}}\colon A \in \mathfrak{I}_x\}.$$

The following is easily verified.

Proposition 14.3. *Let $x \in [\kappa]^\omega$ be $P_{\mathfrak{I}}$-generic over V. Then H_x is generic for $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$ over V. Moreover, $\mathfrak{I}_x = \mathfrak{I}_{H_x}$.*

Assume that $add(\mathfrak{I}) = cov(\mathfrak{I})$. Given $H \subseteq \mathfrak{I}^+/\mathfrak{I}$ such that H is generic for $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$ over V, we set $\mathfrak{I}'_H = \{C \in V \cap P(add(\mathfrak{I}))\colon \bigcup_{\beta \in C} R_\beta \in \mathfrak{I}_H\}$.

Notice that $\mathfrak{I}'_H = \mathfrak{I}_H$ in case $add(\mathfrak{I}) = \kappa$.

It was mentioned in Section 0 that assuming $\kappa > \omega$, the existence of a κ-distributive ideal J over κ verifying $add(J) = \kappa$ does not necessarily entail the measurability of κ. Such a J will however have the property that κ is measurable in any extension obtained by forcing with $(J^+/J, \leqslant)$, as (the well-known) Proposition 14.4 shows.

Proposition 14.4. *Assume that $add(\mathfrak{I}) = cov(\mathfrak{I})$ and $\mathfrak{I}$ is $add(\mathfrak{I})$-distributive. Let $H \subseteq \mathfrak{I}^+/\mathfrak{I}$ be generic for $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$ over V. Then $\mathfrak{I}'_H$ is a prime ideal over $add(\mathfrak{I})$ such that $add(\mathfrak{I}) \subseteq \mathfrak{I}'_H \subset P\ (add(\mathfrak{I}))$ and $add(\mathfrak{I}'_H) = add(\mathfrak{I})$.*

Proof. Given $f \in V[H]$ with $f\colon add(\mathfrak{I}) \to V$, we have $f \in V$, as $\mathfrak{I}$ is $add(\mathfrak{I})$-distributive. Hence by Proposition 14.1, $add(\mathfrak{I}_H) = add(\mathfrak{I})$. The desired properties of $\mathfrak{I}'_H$ are now easily verified. $\square$

Proposition 14.5. *Assume that $add(\mathfrak{I}) = cov(\mathfrak{I})$ and $\mathfrak{I}$ is almost κ-distributive. Let $x \in [\kappa]^\omega$ be $P_{\mathfrak{I}}$-generic over V. Then t_x is $P_{\mathfrak{I}'_{H_x}}$-generic over $V[H_x]$.*

Proof. First observe that by Proposition 3.3, $\mathfrak{I}$ is $add(\mathfrak{I})$-distributive. So Proposition 14.4 applies. Let $k\colon \kappa \to add(\mathfrak{I})$ be such that for every $\alpha \in \kappa$, $\alpha \in R_{k(\alpha)}$. Define $F\colon [\kappa]^2 \to 2$ by letting $F(a) = 0$ if and only if $k(\bigcap a) \geqslant k(\bigcup a)$. By Proposition 13.3, there is a $\delta \in x$ such that F is constant on $[x - \delta]^2$. It easily follows from Corollary 11.3 that F is identically 1 on $[x - \delta]^2$. Hence, by Proposition 12.6, $t_x \in [add(\mathfrak{I})]^\omega$. Let $F'\colon [add(\mathfrak{I})]^2 \to 2$ be given in $V[H_x]$. By $add(\mathfrak{I})$-distributivity of $\mathfrak{I}$, $F' \in V$. Define, in V, $F''\colon [\kappa]^2 \to 2$ by letting $F''(a) = 0$ if and only if $k(\bigcap a) \neq k(\bigcup a)$ and $F'(\{k(\bigcap a), k(\bigcup a)\}) = 0$. By Proposition 13.3, there is an $\eta \in x$ such that F'' is constant on $[x - \eta]^2$. Select $\xi \in x - \delta$ so that for every $\alpha \in x \cap \delta$, $k(\alpha) < k(\xi)$. Clearly, F' is constant on $[t_x - k(\xi \cup \eta)]^2$. Thus, t_x is Ramsey over V. Now let $C \in \mathfrak{I}'_{H_x}$. As by Proposition 14.3, $\mathfrak{I}_x = \mathfrak{I}_{H_x}$, we have $\bigcup_{\beta \in C} R_\beta \in \mathfrak{I}_x$. Thus $x \cap \bigcup_{\beta \in C} R_\beta \in [\kappa]^{<\omega}$, and consequently $t_x \cap C \in [add(\mathfrak{I})]^{<\omega}$. As $\mathfrak{I}'_{H_x}$ is prime by Proposition 14.4, this shows that for every $Q \in V[H_x]$ with $Q \in M_{\mathfrak{I}'_{H_x}, add(\mathfrak{I})}$, there is a $D \in Q$ with $t_x - D \in [add(\mathfrak{I})]^{<\omega}$. Hence by Propositions 14.4 and 13.5, t_x is $P_{\mathfrak{I}'_{H_x}}$-generic over $V[H_x]$. $\square$

Proposition 14.6. *Assume that $\mathfrak{I}$ is κ-distributive and $add(\mathfrak{I}) = \kappa$. Let $H \subseteq \mathfrak{I}^+/\mathfrak{I}$ be generic for $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$ over V, and let $x \in [\kappa]^\omega$ be $P_{\mathfrak{I}_H}$-generic over $V[H]$. Then x is $P_{\mathfrak{I}}$-generic over V.*

Proof. Clearly, $add(\mathfrak{I}) = cov(\mathfrak{I}) = \kappa$, and so Proposition 14.4 applies. Moreover, $\mathfrak{I}'_H = \mathfrak{I}_H$. Hence, $\mathfrak{I}_H$ is a prime ideal over κ with $add(\mathfrak{I}_H) = \kappa$. By Proposition 14.1, we have $\mathfrak{I} \subseteq \mathfrak{I}_H$. Since x is $P_{\mathfrak{I}_H}$-generic over $V[H]$, we have by Proposition 13.3 that x is Ramsey over V. Now let, in V, $Q \in M_{\mathfrak{I},\kappa}$, and set $X = \{[B]_{\mathfrak{I}}: B \in \mathfrak{I}^+ \cap \bigcup_{A \in Q} P(A)\}$. X is clearly dense in $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$ and, therefore, $X \cap H \neq 0$. Hence $\mathfrak{I}_H^* \cap \bigcup_{A \in Q} P(A) \neq 0$. As x is $P_{\mathfrak{I}_H}$-generic over $V[H]$, we have by Corollary 11.3 that for all $B \in \mathfrak{I}_H^*$, $x - B \in [\kappa]^{<\omega}$. Then there is an $A \in Q$ with $x - A \in [\kappa]^{<\omega}$. Hence, by Proposition 13.5, x is $P_{\mathfrak{I}}$-generic over V. □

15. $G_{\mathfrak{I}}^{**}(a, C, W)$

We now briefly consider a new game that will be used in the next section, where it will be assumed that $\kappa = \omega$. Since however the two results we need do not depend on that assumption, we will state them for any arbitrary κ.

Given $a \in [\kappa]^{<\omega}$, $C \in \mathfrak{I}^+$ and $W \subseteq [\kappa]^\omega$, we define the two-person game $G_{\mathfrak{I}}^{**}(a, C, W)$ as follows:

First I chooses $A_0 \in \mathfrak{I}^+ \cap P(C)$, II responds with a choice of $\alpha_0 \in \{\beta \in A_0: a \subseteq \beta\}$; then I chooses $A_1 \in \mathfrak{I}^+ \cap P(A_0 - (\alpha_0 + 1))$, II selects $\alpha_1 \in A_1$, and so on up to ω.

I wins if $a \cup \{\alpha_i: i \in \omega\} \in W$.

Proposition 15.1. *Let $a \in [\kappa]^{<\omega}$, $C \in \mathfrak{I}^+$ and $W \subseteq [\kappa]^\omega$ be given, and assume that* I *has a winning strategy in $G_{\mathfrak{I}}^*(a, C, W)$. Then* II *has a winning strategy in $G_{\mathfrak{I}}^{**}(a, D, [\kappa]^\omega - W)$ for some $D \in \mathfrak{I}^+ \cap P(C)$.*

Proof. Let σ be a winning strategy for I in $G_{\mathfrak{I}}^*(a, C, W)$. We define a winning strategy τ for II in $G_{\mathfrak{I}}^{**}(a, \sigma(0), [\kappa]^\omega - W)$ by letting $\tau(A_0) = \bigcap \{\beta \in A_0: a \subseteq \beta\}$, and $\tau(A_0, \ldots, A_{i+1}) = \bigcap (A_{i+1} \cap \sigma(\tau(A_0), \ldots, \tau(A_0, \ldots, A_i)))$. □

Notice that $G_{\mathfrak{I}}^{**}(a, C, W)$ and $G_{\mathfrak{I}}^*(a, C, W)$ are identical in case $\mathfrak{I}$ is prime. It thus follows from Proposition 15.1 that assuming $\mathfrak{I}$ to be prime, the existence of a winning strategy for I in $G_{\mathfrak{I}}^{**}(a, C, W)$ implies the existence of a winning strategy for II in $G_{\mathfrak{I}}^{**}(a, D, [\kappa]^\omega - W)$ for some $D \in \mathfrak{I}^+ \cap P(C)$. We remark that this need not hold in general. In fact, we will see in the next section (see Proposition 16.2 and the proof of Proposition 16.11) that there is an ideal K over ω that satisfies the following: I has a winning strategy in $G_K^{**}(0, \omega, [\omega]^\omega - K^+)$ but for every $D \in K^+$, II has no winning strategy in $G_K^{**}(0, D, K^+)$. On the other hand, one can in all cases define a winning strategy for player I from one for player II:

Proposition 15.2. *Let $a \in [\kappa]^{<\omega}$, $C \in \mathfrak{I}^+$ and $W \subseteq [\kappa]^\omega$ be given, and assume that* II *has a winning strategy in $G_{\mathfrak{I}}^{**}(a, C, W)$. Then* I *has a winning strategy in $G_{\mathfrak{I}}^{**}(a, C, [\kappa]^\omega - W)$.*

Proof. Let τ be a winning strategy for II in $G_{\mathfrak{I}}^{**}(a, C, W)$. We will define a winning strategy σ for I in $G_{\mathfrak{I}}^{**}(a, C, [\kappa]^\omega - W)$. We put $\sigma(0) = \{\tau(A): A \in \mathfrak{I}^+ \cap P(C)\}$.

Assume II's successive moves are α_i for $i < \omega$. Define A_i and B_i for $i < \omega$ so that

(0) $B_0 = \sigma(0)$;

(1) $B_{i+1} = \{\tau(A_0, \ldots, A_i, A)\colon A \in \mathfrak{I}^+ \cap P(A_i \cap B_i)\}$;

(2) $A_0 \in \mathfrak{I}^+ \cap P(C)$;

(3) $A_{i+1} \in \mathfrak{I}^+ \cap P(A_i \cap B_i)$;

(4) $\alpha_i = \tau(A_0, \ldots, A_i)$.

Then set $\sigma(\alpha_0, \ldots, \alpha_i) = B_{i+1}$. $\square$

16. Ideals over ω

Throughout this section we assume that $\kappa = \omega$.

We are going to see that some standard combinatorial properties of ideals have equivalent formulations in terms of the existence (or the non-existence) of winning strategies. We start with the following.

Proposition 16.1. (i) *$\mathfrak{I}$ is a weak Q-point if and only if* I *has no winning strategy σ in $G^*_{[\omega]^{<\omega}}(0, \omega, \mathfrak{I} \cap [\omega]^\omega)$ such that $\sigma(0) \in \mathfrak{I}^+$.*

(ii) *$\mathfrak{I}$ is weakly selective if and only if* I *has no winning strategy in $G^*_3(0, \omega, \mathfrak{I} \cap [\omega]^\omega)$.*

Proof. To prove (ii), we will use Lemma 1.8. First given $A \in \mathfrak{I}^+$ and $H_n \in \mathfrak{I}$ for $n < \omega$, we define a strategy σ for I in $G^*_3(0, \omega, [\omega]^\omega)$ by letting $\sigma(0) = A$ and $\sigma(n_0, \ldots, n_i) = (A - \bigcup_{j \leqslant i} H_{n_j}) - (n_i + 1)$. If σ is not a winning strategy for I in $G^*_3(0, \omega, \mathfrak{I} \cap [\omega]^\omega)$, then clearly there is a $B \in \mathfrak{I}^+ \cap P(A)$ such that $m \notin \bigcup_{n \in B \cap m} H_n$ for all $m \in B$.

Next let us assume that I has a winning strategy τ in $G^*_3(0, \omega, \mathfrak{I} \cap [\omega]^\omega)$. For each $n \in \omega$, we put $K_n = \bigcup \{F_\tau(b)\colon\ b \in P(n+1) - \{0\}\}$. Then clearly there is no $D \in \mathfrak{I}^+ \cap \tau(0)$ such that $m \notin \bigcup_{n \in D \cap m} K_n$ for all $m \in D$.

The proof of (i) is left to the reader, as it is a straightforward modification of that of (ii). $\square$

Proposition 16.2. *The following are equivalent:*

(i) *$\mathfrak{I}$ is $\aleph_0$-distributive and weakly selective.*

(ii) *Given $C \in \mathfrak{I}^+$,* II *has no winning strategy in $G_3(0, C, \mathfrak{I}^+)$.*

(iii) *$\mathfrak{I}$ is $\aleph_0$-distributive and given $C \in \mathfrak{I}^+$,* II *has no winning strategy in $G^{**}_3(0, C, \mathfrak{I}^+)$.*

(iv) *$\mathfrak{I}^+ \cap P(A) \notin N_3$ for every $A \in \mathfrak{I}^+$.*

(v) I *has no winning strategy in $G_3(0, \omega, \mathfrak{I} \cap [\omega]^\omega)$.*

Proof. (i) $\to$ (ii): Assume that (i) holds and there is a $C \in \mathfrak{I}^+$ such that II has a winning strategy in $G_3(0, C, \mathfrak{I}^+)$. Now by Proposition 7.1, I has a winning strategy σ in $G_3(0, C, [\omega]^\omega - \mathfrak{I}^+)$. Then $\langle 0, \varepsilon_3(\varepsilon_2(\varepsilon_1(\sigma)))\rangle \cap \mathfrak{I}^+ = 0$, an obvious contradiction.

(ii) $\to$ (iii): Assume (ii). Then clearly for every $C \in \mathfrak{I}^+$, II has no winning strategy in $G^{**}_3(0, C, \mathfrak{I}^+)$. Let us show that $\mathfrak{I}$ is $\aleph_0$-distributive. Thus let $Q_n \in M_{\mathfrak{I},\omega}$ for $n < \omega$, and let $C \in \mathfrak{I}^+$. Let W be the set of all $D \in [\omega]^\omega$ such that for every $n \in \omega$, there is a $B \in Q_n$ with $D - B \in [\omega]^{<\omega}$. We will define a winning strategy τ for II in $G_{\mathfrak{I}}(0, C, [\omega]^\omega - W)$. We let $\tau(A_0, \ldots, A_i) = (n_i, (A_i \cap B_i) - (n_i + 1))$, where $B_i \in Q_i$ is such that $A_i \cap B_i \in \mathfrak{I}^+$, and $n_i = \bigcap (A_i \cap B_i)$. Now τ is not a winning strategy for II in $G_{\mathfrak{I}}(0, C, \mathfrak{I}^+)$. Hence $\mathfrak{I}^+ \cap P(C) \cap W \neq 0$.

(iii) → (iv): Let us assume that $\mathfrak{I}$ is $\aleph_0$-distributive and there is an A with $\mathfrak{I}^+ \cap P(A) \in N_{\mathfrak{I}}$. Let σ be a winning strategy for I in $G_{\mathfrak{I}}(0, A, [\omega]^\omega - (\mathfrak{I}^+ \cap P(A)))$. Then $\varepsilon_1(\sigma)$ is a winning strategy for I in $G^*_{\mathfrak{I}}(0, \omega, \mathfrak{I} \cap [\omega]^\omega)$. Applying Proposition 15.1, we obtain that II has a winning strategy in $G^{**}_{\mathfrak{I}}(0, C, \mathfrak{I}^+)$ for some $C \in \mathfrak{I}^+$.

(iv) → (v): Assume that I has a winning strategy σ in $G_{\mathfrak{I}}(0, \omega, \mathfrak{I} \cap [\omega]^\omega)$. Given $a \in [\omega]^{<\omega}$ and $C \in \mathfrak{I}^+$, we define a winning strategy σ' for I in $G_{\mathfrak{I}}(a, C, [\omega]^\omega - (\mathfrak{I}^+ \cap P(\sigma(0))))$ by letting $\sigma'(0) = \sigma(0)$ and $\sigma'((\alpha_0, B_0), \ldots, (\alpha_i, B_i)) = \sigma((\alpha_0, B_0), \ldots, (\alpha_i, B_i))$ in case $C \cap \sigma(0) \in \mathfrak{I}^+$, and $\sigma' = \ulcorner C - \sigma(0) \urcorner$ otherwise. Thus $\mathfrak{I}^+ \cap P(\sigma(0)) \in N_{\mathfrak{I}}$.

(v) → (i): Assume (v). Let $Q_n \in M_{\mathfrak{I},\omega}$ for $n < \omega$, and let $C \in \mathfrak{I}^+$. Let W be the set of all $D \in [C]^\omega$ such that for every $n \in \omega$, there is a $B \in Q_n$ with $D - B \in [\omega]^{<\omega}$. Define $\sigma \in \Sigma^\omega_{\mathfrak{I}}$ so that $\sigma(0) = C$ and $\sigma((\alpha_0, B_0), \ldots, (\alpha_i, B_i)) \in \bigcup_{B \in Q_i} P(B)$. Then σ is a winning strategy for I in $G_{\mathfrak{I}}(0, \omega, W)$. By (iv) $W \cap \mathfrak{I}^+ \neq 0$. Thus, $\mathfrak{I}$ is $\aleph_0$-distributive. Now let $B \in \mathfrak{I}^+$, and let $g : B \to \omega$ be such that $g^{-1}(\{n\}) \in \mathfrak{I}$ for all $n \in \omega$. Let Y be the set of all $D \in [B]^\omega$ such that g is one-to-one on D. Define $\sigma \in \Sigma^\omega_{\mathfrak{I}}$ by letting $\sigma(0) = B$ and $\sigma((\alpha_0, B_0), \ldots, (\alpha_i, B_i)) = B_i - g^{-1}(\{g(\alpha_i)\})$. Then σ is a winning strategy for I in $G_{\mathfrak{I}}(0, \omega, Y)$. By (iv), $Y \cap \mathfrak{I}^+ \neq 0$. Thus $\mathfrak{I}$ is weakly selective. □

Clearly, $\mathfrak{I}^+ \in C_{[\omega]^{<\omega}}$. Notice however that $\mathfrak{I}^+ \in N_{[\omega]^{<\omega}}$ if and only if $\mathfrak{I}$ is tall. Here is another fact concerning tall ideals.

Proposition 16.3. *The following are equivalent:*

(i) $\mathfrak{I}^+ \notin C_{\mathfrak{I}}$.

(ii) *There is an* $A \in \mathfrak{I}^+$ *such that* $\mathfrak{I} | A$ *is tall,* $\aleph_0$*-distributive and weakly selective.*

Proof. (i) → (ii): Assume (ii) fails. Let $A \in \mathfrak{I}^+$. Suppose $\mathfrak{I} | A$ is not tall. Then there is a $D \in \mathfrak{I}^+ \cap P(A)$ with $[D]^\omega \subseteq \mathfrak{I}^+$. Therefore, I has a winning strategy in $G_{\mathfrak{I}}(a, A, \mathfrak{I}^+)$ for every $a \in [\omega]^{<\omega}$. Now suppose that $\mathfrak{I} | A$ is either not $\aleph_0$-distributive, or not weakly selective. Then by Proposition 16.2, I has a winning strategy in $G_{\mathfrak{I}|A}(0, \omega, (\mathfrak{I} | A) \cap [\omega]^\omega)$. Then clearly I has a winning strategy in $G_{\mathfrak{I}}(a, A, \mathfrak{I} \cap [\omega]^\omega)$ for every $a \in [\omega]^{<\omega}$. Thus $\mathfrak{I}^+ \in C_{\mathfrak{I}}$.

(ii) → (i): Let A be as in (ii). Then for every $B \in (\mathfrak{I} | A)^+ \cap P(A)$, we have $\langle 0, B \rangle - \mathfrak{I}^+ \neq 0$ and $\langle 0, B \rangle \cap \mathfrak{I}^+ \neq 0$. Hence by Proposition 10.23, $\mathfrak{I}^+ \notin C_{\mathfrak{I}|A}$. As it is readily checked that $C_{\mathfrak{I}} \subseteq C_{\mathfrak{I}|A}$, we obtain $\mathfrak{I}^+ \notin C_{\mathfrak{I}}$. □

Corollary 16.4. *Assuming* $\mathfrak{I}$ *is prime, the following are equivalent:*

(i) $\mathfrak{I}^+ \notin C_{\mathfrak{I}}$.

(ii) $\mathfrak{I}^+ \notin N_{\mathfrak{I}}$.

(iii) $\mathfrak{I}$ *is weakly selective.*

Proof. (i) → (ii): Clear.

(ii) → (iii): Use Proposition 16.1(ii) and Proposition 9.3.

(iii) → (i): By Proposition 16.3, since $\mathfrak{I}$ is tall, $\aleph_0$-distributive and such that for every $A \in \mathfrak{I}^+$, $\mathfrak{I} | A = \mathfrak{I}$. □

We will now study some properties of nowhere tall ideals.

Proposition 16.5. *The following are equivalent*:

(i) *For every $C \in \mathfrak{I}^+$, I has a winning strategy in $G^*_{\mathfrak{I}}(0, C, \mathfrak{I}^+)$.*

(ii) *$\mathfrak{I}$ is nowhere tall.*

(iii) *$\mathfrak{I} \cap [\omega]^\omega \in N_{\mathfrak{I}}$.*

(iv) *$\mathfrak{I}$ is $\aleph_0$-distributive and weakly selective and, moreover, $\mathfrak{I}^+ \in C_{\mathfrak{I}}$.*

(v) *For every $C \in \mathfrak{I}^+$, I has a winning strategy in $G^*_{[\omega]^{<\omega}}(0, C, \mathfrak{I}^+)$.*

Proof. (i) → (ii): Let $C \in \mathfrak{I}^+$, and let σ be a winning strategy for I in $G^*_{\mathfrak{I}}(0, C, \mathfrak{I}^+)$. We define $f: \omega \to \mathfrak{I}$ by letting $f(p) = \bigcup \{F_\sigma(b): b \in P(p+1) - \{0\}\}$. By induction define p_i for $i < \omega$ so that $p_0 \in \sigma(0)$ and for every $i, p_{i+1} \in \sigma(0) - f(p_i)$. Clearly, $[\{p_i: i < \omega\}]^\omega \subseteq \mathfrak{I}^+$.

(ii) → (iii): Straightforward.

(iii) → (iv): Assume (iii). Then for every $C \in \mathfrak{I}^+$, I has a winning strategy in $G_{\mathfrak{I}}(0, C, \mathfrak{I}^+)$. Hence by Proposition 16.2, $\mathfrak{I}$ is $\aleph_0$-distributive and weakly selective. As $N_{\mathfrak{I}} \subseteq C_{\mathfrak{I}}$, we have $\mathfrak{I} \cap [\omega]^\omega \in C_{\mathfrak{I}}$, or equivalently $\mathfrak{I}^+ \in C_{\mathfrak{I}}$.

(iv) → (v): Assume (iv). Given $C \in \mathfrak{I}^+$, there is by Proposition 10.23 a $B \in \mathfrak{I}^+ \cap P(C)$ such that either $\langle 0, B \rangle \subseteq \mathfrak{I}^+$, or else $\langle 0, B \rangle \cap \mathfrak{I}^+ = 0$. Clearly, $[B]^\omega \subseteq \mathfrak{I}^+$. Hence I has a winning strategy in $G^*_{[\omega]^{<\omega}}(0, C, \mathfrak{I}^+)$.

(v) → (i): Clear. □

Corollary 16.6. *Assume that $\mathfrak{I}|A$ is tall for some $A \in \mathfrak{I}^+$. Then $add(N_{\mathfrak{I}}) \leqslant cof(\mathfrak{I})$.*

Proof. Let $K \subseteq \mathfrak{I}$ be such that $\mathfrak{I} = \bigcup_{B \in K} P(B)$. Clearly $[B]^\omega \in N_{\mathfrak{I}}$ for each $B \in K$. By Proposition 16.5 $\bigcup_{B \in K} [B]^\omega \notin N_{\mathfrak{I}}$. □

The proof of the following makes use of the easy fact that $[A]^\omega \in N_{\mathfrak{I}} - N_{[\omega]^{<\omega}}$ whenever $A \in \mathfrak{I} \cap [\omega]^\omega$.

Proposition 16.7. *Assume that $\mathfrak{I}$ is nowhere tall and $\mathfrak{I} \neq [\omega]^{<\omega}$. Then $N_{[\omega]^{<\omega}} \subset N_{\mathfrak{I}}$ and $C_{[\omega]^{<\omega}} \subset C_{\mathfrak{I}}$.*

Proof. It is readily checked that $N_{[\omega]^{<\omega}} \subset N_{\mathfrak{I}}$ and $C_{[\omega]^{<\omega}} \subseteq C_{\mathfrak{I}}$. Now let $A \in \mathfrak{I} \cap [\omega]^\omega$. By Corollary 6.5, there is a $W \subseteq [A]^\omega$ such that $W \notin C_{[\omega]^{<\omega}}$. Clearly, W belongs to $N_{\mathfrak{I}}$ and therefore to $C_{\mathfrak{I}}$. □

We observe that if $\mathfrak{I}$ is nowhere tall, then by Propositions 2.4, 2.10, 2.13, 5.12, 10.15 and 10.21, $add(N_{\mathfrak{I}})$, $cov(N_{\mathfrak{I}})$, $non(N_{\mathfrak{I}})$ and $cof(N_{\mathfrak{I}})$ are respectively the same as $add(N_{[\omega]^{<\omega}})$, $cov(N_{[\omega]^{<\omega}})$, $non(N_{[\omega]^{<\omega}})$ and $cof(N_{[\omega]^{<\omega}})$.

Given two properties of the type considered in this paper, one may ask whether there is an ideal that satisfies one but not the other. We will now deal with that kind of question. For that we have to consider specific ideals. We first introduce some notation.

Given $Q \subseteq [\omega]^{\omega}$, we let J_Q be the set of all $A \subseteq \omega$ such that $A \cap B \in [\omega]^{<\omega}$ for all $B \in Q$.

The following is straightforward.

Lemma 16.8. *Given $Q \subseteq [\omega]^{\omega}$, J_Q is a nowhere tall ideal over ω.*

We observe that given $Q \in D_{[\omega]^{<\omega},\omega}$, $J_Q = [\omega]^{<\omega}$ if and only if $Q \in M_{[\omega]^{<\omega},\omega}$.

Given $Q \in D_{[\omega]^{<\omega},\omega}$, we let K_Q be the set of all $C \subseteq \omega$ such that for some $X \in [Q]^{<\omega}$, $C - \bigcup X \in [\omega]^{<\omega}$.

Clearly, K_Q is an ideal over ω. Let us assume that $\omega \notin K_Q$. Then by Proposition 3.2 of [23], K_Q is weakly selective and such that $(K_Q^+/K_Q, \leqslant)$ is $\aleph_0$-closed. Moreover, K_Q is easily seen to be everywhere feeble. We also have the following.

Proposition 16.9. *Let $Q \in D_{[\omega]^{<\omega},\omega}$ be such that $\omega \notin K_Q$. Then $\boldsymbol{k}_{K_Q} = \boldsymbol{h}_{K_Q}$.*

Proof. An easy modification of the proof of Proposition 13.1 in [23]. □

Let us also mention the following, which could possibly be improved (by replacing $\boldsymbol{b}$ by a larger cardinal invariant).

Proposition 16.10. *Let Q be an infinite member of $M_{[\omega]^{<\omega},\omega}$. Then $\pi_{K_Q} \geqslant \boldsymbol{b}$.*

Proof. Let $C \in K_Q^+$, and let $Y \subseteq Q$ with $0 < |Y| < \boldsymbol{b}$. By Proposition 3.3 of [23], there is a one-to-one $w: \omega \to Q - Y$ such that $\{C \cap w(n): n \in \omega\} \subseteq [\omega]^{\omega}$. For each $B \in Y$, define $f_B \in \omega^{\omega}$ by letting $f_B(n) = \bigcup (B \cap w(n))$. Select $g \in \omega^{\omega}$ so that for every $B \in Y$, $\{n \in \omega\colon\ g(n) \leqslant f_B(n)\} \in [\omega]^{<\omega}$. Then set $D = \bigcup_{n \in \omega}((C \cap w(n)) - g(n))$. Clearly, $D \in K_Q^+ \cap P(C)$. Moreover, $D \cap B \in [\omega]^{<\omega}$ for all $B \in Y$. The desired conclusion easily follows. □

Proposition 16.11. *Consider the following assertions:*

(i) I *has no winning strategy in* $G_{\mathfrak{I}}^{**}(0, \omega, \mathfrak{I} \cap [\omega]^{\omega})$.

(ii) $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$ *is $\aleph_0$-closed, and $\mathfrak{I}$ is weakly selective.*

(iii) *$\mathfrak{I}$ is $\aleph_0$-distributive and weakly selective.*

(iv) *$\mathfrak{I}$ is an $\aleph_0$-distributive weak P-point (respectively weak Q-point).*

(v) *$\mathfrak{I}$ is $\aleph_0$-distributive.*

Then (i) → (ii) → (iii) → (iv) → (v), *and none of those implications can be reversed.*

Proof. (i) $\to$ (ii): Assume (i). Let $C_n \in \mathfrak{I}^+$ for $n < \omega$ be such that $C_{n+1} \subseteq C_n$. Let W be the set of all $B \in [\omega]^\omega$ such that for all $n \in \omega$, $B - C_n \in [\omega]^{<\omega}$. We define a winning strategy σ for I in $G_{\mathfrak{I}}^{**}(0, \omega, W)$ by letting $\sigma(0) = C_0$ and $\sigma(n_0, \ldots, n_i) = C_{i+1}$. Since, σ is not a winning strategy for I in $G_{\mathfrak{I}}^{**}(0, \omega, \mathfrak{I} \cap [\omega]^\omega)$, we have $W \cap \mathfrak{I}^+ \neq 0$. Thus, $(\mathfrak{I}^+/\mathfrak{I}, \leqslant)$ is $\aleph_0$-closed. $\mathfrak{I}$ is weakly selective by Proposition 16.2.

Let us now show that (ii) does not imply (i). Select a strategy σ for I in $G_{[\omega]^{<\omega}}^{**}(0, \omega, [\omega]^\omega)$ such that the following holds:

(0) $\sigma(n) \cap \sigma(p) = 0$ for all $n, p \in \sigma(0)$ with $n \neq p$;

(1) let $k \in \omega - 1$, and let n_i for $i < k$ be such that $n_0 \in \sigma(0)$ and for every i with $0 < i < k$, $n_i \in \sigma(n_0, \ldots, n_{i-1})$. Then $\sigma(n_0, \ldots, n_{k-1}, p) \cap \sigma(n_0, \ldots, n_{k-1}, n) = 0$ for all $n, p \in \sigma(n_0, \ldots, n_{k-1})$ with $n \neq p$.

We define $Q \subseteq [\omega]^\omega$ by letting $A \in Q$ if and only if $e_A(0) \in \sigma(0)$ and for all $i \in \omega - 1$, $e_A(i) \in \sigma(e_A(0), \ldots, e_A(i-1))$. Clearly, $Q \in D_{[\omega]^{<\omega}, \sigma(0)}$. Now let k $\in \omega$, and let $z \in \omega^k$ be such that $z(0) \in \sigma(0)$ and for every i with $0 < i < k$, $z(i) \in \sigma(z \mid i)$. We claim that $\sigma(z) \in K_Q^+$. Thus let $r \in \omega - 1$ and $A_j \in Q$ for $j < r$. Select $m_p \in \omega$ for $p < \omega$ so that

(a) $m_p < m_{p+1}$;

(b) $\{m_p\colon p < \omega\} \in Q$;

(c) $\{m_p\colon p < k\} = ran(z)$;

(d) $m_k \notin \{e_{A_j}(k)\colon j < r\}$.

It is easily verified that $\{m_p\colon p < \omega\} \cap \bigcup_{j<r} A_j \subseteq ran(z)$. Hence, $\sigma(z) - \bigcup_{j<r} A_j \in [\omega]^\omega$. By the claim, σ is a winning strategy for I in $G_{K_Q}^{**}(0, \omega, Q)$. Hence, I has a winning strategy in $G_{K_Q}^{**}(0, \omega, K_Q \cap [\omega]^\omega)$.

(ii) $\to$ (iii): Easy.

Let us show that (iii) does not imply (ii). Let $Q \subset [\omega]^\omega$ be such that $|Q| = \aleph_0$, and that $A \cap B = 0$ for all $A, B \in Q$ with $A \neq B$. By Lemma 16.8 and Proposition 16.5, J_Q is $\aleph_0$-distributive and weakly selective. It remains to show that $(J_Q^+/J_Q, \leqslant)$ is not $\aleph_0$-closed. Let B_n for $n < \omega$ be a one-to-one enumeration of Q. For each $p < \omega$, set $C_p = \bigcup_{n \in \omega - p} B_n$. It is clear that each $C_p \in J_Q^+$. Moreover, $C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots$ Now let $C \in J_Q^+$ be given. Select $k \in \omega$ with $C \cap B_k \in [\omega]^\omega$. We have $C - C_{k+1} \in J_Q^+$ and, therefore, it is not the case that $[C]_{J_Q} \leqslant [C_{k+1}]_{J_Q}$.

(iii) $\to$ (iv) $\to$ (v): Trivial.

Let $E_0, E_1 \in [\omega]^\omega$ be such that $E_0 \cap E_1 = 0$ and $E_0 \cup E_1 = \omega$. Select pairwise disjoint $A_n \in [\omega]^\omega$ for $n < \omega$ with $\bigcup_{n \in \omega} A_n = E_0$, and pairwise disjoint $a_n \in [\omega]^{<\omega} - \{0\}$ for $n < \omega$ such that $\bigcup_{n \in \omega} a_n = E_1$ and $\bigcup_{n \in \omega} |a_n| = \aleph_0$. Set

$$J_0 = \{B \subseteq \omega\colon \{n \in \omega\colon B \cap A_n \in [\omega]^\omega\} \in [\omega]^{<\omega}\},$$

$$J_1 = \left\{B \subseteq \omega\colon \bigcup_{n \in \omega} |B \cap a_n| < \aleph_0\right\}$$

and $J_2 = J_0 \cap J_1$. For each $i \leqslant 2$, J_i is easily seen to be an (everywhere feeble) ideal over ω. Moreover, J_0 (respectively J_1) fails to be a weak P-point (resp. a weak Q-point). As for J_2, it is neither a weak P-point nor a weak Q-point. We are going to show that $(J_i^+/J_i, \leqslant)$ is $\aleph_0$-closed for each $i \leqslant 2$, and that J_0 (respectively J_1) is a weak Q-point (resp. a weak P-point).

Let us first deal with J_0. Thus, let $C_m \in J_0^+$ for $m < \omega$ be such that $C_{m+1} \subseteq C_m$. For each $m < \omega$, let $X_m = \{m \in \omega\colon\ C_m \cap A_m \in [\omega]^\omega\}$. Pick $q_i \in X_0$ for $i < \omega$ so that $q_i < q_{i+1}$ and $q_{i+1} \in X_{q_i}$. Set $D = \bigcup_{i<\omega}(C_{q_i} \cap A_{q_{i+1}})$. Clearly, $D \in J_0^+$ and for all $m < \omega$, $D - C_m \in J_0$. Now let $B \in J_0^+$, and let $f\colon B \to \omega$ be such that $|f^{-1}(\{m\})| < \aleph_0$ for each $m \in \omega$. Let $T = \{n \in \omega\colon\ B \cap A_n \in [\omega]^\omega\}$. Select $g\colon \omega \to T$ so that $|g^{-1}(\{n\})| = \aleph_0$ for every $n \in T$. Define $h\colon \omega \to B$ and $k\colon \omega \to \omega$ so that k is one-to-one and $h(m) \in A_{g(m)} \cap f^{-1}(\{k(m)\})$. Setting $E = ran(h)$, we have $E \in J_0^+ \cap P(B)$ and f is one-to-one on E.

Let us now turn to J_1. First let $L_m \in J_1^+$ for $m < \omega$ be such that $L_{m+1} \subseteq L_m$. Define $w\colon \omega \to \omega$ so that w is one-to-one and $|a_{w(m)} \cap L_m| \geqslant m + 1$. Put $H = \bigcup_{m \in \omega}(a_{w(m)} \cap L_m)$. Then $H \in J_1^+$ and for all $m \in \omega$, $H - L_m \in [\omega]^{<\omega}$. Next let $A \in J_1^+$ and $c\colon A \to \omega$ be such that $c^{-1}(\{m\}) \in J_1$ for all $m \in \omega$. Let $k\colon \omega \to \omega$ be such that $|c^{-1}(\{m\}) \cap a_n| \leqslant k(m)$ for all $n, m \in \omega$. Then define $l\colon \omega \to [\omega]^{<\omega}$, $d\colon \omega \to [\omega]^{<\omega}$ and $s\colon \omega \to \omega$ so that

(0) $|d(p)| \geqslant p + 1$;
(1) $d(p) \subseteq \bigcup_{m \in l(p)} c^{-1}(\{m\})$;
(2) $p' < p$ implies $l(p') \cap l(p) = 0$;
(3) s is one-to-one;
(4) $d(p) \subseteq a_{s(p)}$;
(5) $|A \cap a_{s(p+1)}| \geqslant p + 2 + \sum_{i \in \bigcup_{q \leqslant p} l(q)} k(i)$.

Then set $R = \bigcup_{p \in \omega} d(p)$. Clearly $R \in J_1^+$ and for every $m \in \omega$, $|R \cap c^{-1}(\{m\})| < \aleph_0$.

Let us finally show that $(J_2^+/J_2, \leqslant)$ is $\aleph_0$-closed. Thus, let $K_m \in J_2^+$ for $m < \omega$ be such that $K_{m+1} \subseteq K_m$. As $J_2^+ = J_0^+ \cup J_1^+$, there is a $j < 2$ such that for all $m \in \omega$, $K_m \in J_j^+$. Since, $(J_j^+/J_j, \leqslant)$ is $\aleph_0$-closed, there is a $D \in J_j^+$ such that $D - K_m \in J_j$ for all $m \in \omega$. Clearly, $D \cap E_j \in J_j^+$, and for each $m \in \omega$, $(D \cap E_j) - K_m \in J_2$. □

Let us observe the following. If $cof(\mathfrak{I}) = \aleph_0$, then II is easily seen to have a winning strategy in $G_3^{**}(0, \omega, \mathfrak{I} \cap [\omega]^\omega)$. Now assume that $\mathfrak{I}$ is prime and weakly selective. Then by Corollary 16.4, $\mathfrak{I} \cap [\omega]^\omega \notin C_3$. Hence by Proposition 9.6, there are $a \in [\omega]^{<\omega}$ and $C \in \mathfrak{I}^+$ such that I has no winning strategy in either $G_3^*(a, C, \mathfrak{I} \cap [\omega]^\omega)$ or $G_3^*(a, C, \mathfrak{I}^+)$. It easily follows using Proposition 15.2 that neither player has a winning strategy in $G_3^{**}(0, \omega, \mathfrak{I} \cap [\omega]^\omega)$.

We finally observe that if $\mathfrak{I}$ is weakly selective, $\aleph_0$-distributive, nowhere h_3-distributive and such that $sat(\mathfrak{I} | A) = (2^{\aleph_0})^+$ for every $A \in \mathfrak{I}^+$, then by the results of Section 2, $\mathfrak{I}_\omega$ is weakly selective, $(\aleph_0, 2)$-distributive and nowhere $\aleph_0$-distributive.

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